Ideal-system equivariant embedding (I)

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Contents:

Notations

- Considered C*-algebras $A, B, \ldots$ are separable, ...
- ... except multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $\mathrm{Q}(B) := \mathcal{M}(B)/B$, ...
  as e.g., $\mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\,\mathrm{C_0}(\mathbb{R}_+, B) \subset \mathrm{Q}(SB)$.
- T_0 spaces $X, Y, \ldots$ are second countable.
- $\mathcal{I}(A)$ means the lattice of closed ideals of A.
- $\mathrm{CP}(A, B)$ is the cone of completely positive maps $V \colon A \to B$.

The C^*-algebra A is **exact**, if the functor $B \mapsto A \otimes^{\min} B$ is exact.

Since $C^*(F_\infty) \otimes^{\max} \mathcal{L}(H) = C^*(F_\infty) \otimes^{\min} \mathcal{L}(H)$, this is equivalent to the property that each completely positive map of A into a weakly injective (=WEP) C^*-algebra E is automatically nuclear. In particular, exact A is *sub-nuclear* in the sense of D. Voiculescu.

The C^*-algebra A is **strongly purely infinite** if for every $a_1, a_2 \in A_+$ and $\varepsilon > 0$ there exists $d_1, d_2 \in A$ such that, for $j = 1, 2$,

$$\|d_j^* a_j^2 d_j - a_j^2\| < \varepsilon \quad \text{and} \quad \|d_1^* a_1 a_2 d_2\| < \varepsilon \,.$$

The A is “only” *purely infinite*, i.e., each non-zero element of A is properly infinite, if this holds with $a_1 = a_2$. It is not known if “purely infinite” always implies “strongly purely infinite”. If Prim(A) is Hausdorff or if $\mathcal{I}(A)$ is linearly ordered “p.i.” implies “s.p.i.”

Theorem (Representing KK-classes by morphisms)

*Suppose that A and B are stable and separable, and that $\mathcal{C}_1 \subset \mathrm{CP}(A, B)$ is an m.o.c.c., and that there exists a non-degenerate *-monomorphism $h_1 \colon A \to B$ such that h_1 generates the m.o.c. cone $\mathcal{C}_1$ and that $h_1 \oplus h_1$ is unitarily equivalent to h_1. Then:*

(i) *the natural semi-group morphism from the semi-group of unitary equivalence classes $[\mathrm{Hom}(A, B) \cap \mathcal{C}_1]_u$ into $\mathrm{KK}(\mathcal{C}_1\, ;\, A, B)$ – induced by $\varphi \mapsto [(B, \varphi, 0)]$ – is surjective, and*

(ii) $[\psi] = [\varphi]$ *holds in* $\mathrm{KK}(\mathcal{C}_1\, ;\, A, B)$ *if and only if $\psi \oplus h$ and $\varphi \oplus h$ are unitarily homotopic*
(i.e. if there exists a norm-continuous map
$t \in [0, \infty) \mapsto u(t) \in \mathcal{U}(\mathcal{M}(B))$ *with* $u(0) = 1$ *and*
$\lim_{t\to\infty} u(t)^*(\varphi(a) \oplus h(a))u(t) = \psi(a) \oplus h(a)$ *for $a \in A$).*

Corollary ($\mathcal{C}$-classification)

*If, in addition to the assumptions of the last theorem, $\mathcal{C}_2 \subset \mathrm{CP}(B, A)$ is an m.o.c.c. such that there is non-degenerate *-morphism $h_2 \colon B \to A$ that generates $\mathcal{C}_2$ and is unitarily equivalent to $h_2 \oplus h_2$, then:*
There is an isomorphism φ from A onto B with $\varphi \in \mathcal{C}_1$ and $\varphi^{-1} \in \mathcal{C}_2$, if and only if

(i) $\mathrm{id}_A \in \mathcal{C}_2 \circ \mathcal{C}_1$,

(ii) $\mathrm{id}_B \in \mathcal{C}_1 \circ \mathcal{C}_2$, *and*

(iii) *there are $z_1 \in \mathrm{KK}(\mathcal{C}_1\,;\ A, B)$ and $z_2 \in \mathrm{KK}(\mathcal{C}_2\,;\ B, A)$ with $z_1 \otimes_A z_2 = [\mathrm{id}_B]$ in $\mathrm{KK}(\mathcal{C}_1 \circ \mathcal{C}_2; B, B)$ and $z_2 \otimes_B z_1 = [\mathrm{id}_A]$ in $\mathrm{KK}(\mathcal{C}_2 \circ \mathcal{C}_1; A, A)$.*

If $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the assumptions (i)-(iii) of the Corollary on $\mathcal{C}$-classification, then the isomorphism φ can be found such that $[\varphi] = z_1$ in addition. The φ is unique up to unitary homotopy.

We have seen that the existence of a suitable *-monomorphism $h_0 \colon A \to B$ that represents the zero element of $\mathrm{KK}(\mathcal{C}; A, B)$ plays an important role for the ideal-equivariant classification. The following partial result will be discussed in lectures 2-4:

Theorem (Existence of h_0: The embedding Theorem.)

Suppose that A and B are stable, A is exact and B is strongly purely infinite, and that $\Psi \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ is a non-degenerate action of $\mathrm{Prim}(B)$ on A lower s.c. and monotone upper s.c.

Then there is a non-degenerate nuclear monomorphism $h_0 \colon A \to B$ such that $h_0 \oplus h_0$ is unitarily equivalent to h_0, and $\mathcal{C}(h_0) = \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B)\,;\, A, B) = \mathrm{CP}_{\mathrm{nuc}}(A, B) \cap \mathrm{CP}(\Psi;\, A, B)$.

Thus $[\mathrm{Hom}_{\mathrm{nuc}}(\mathrm{Prim}(B); A, B) \oplus h_0]_{u(t)} \cong \mathrm{KK}(\mathrm{Prim}(B)\,;\, A, B)$.

Notice that a C^*-algebra B is stable, i.e., $B \cong B \otimes \mathbb{K}$ if and only if there exists a sequence of isometries $s_n \in \mathcal{M}(B)$ with the property $\sum_n s_n s_n^* = 1$ (with strict convergence).

Then the **infinite repeat** $\delta_\infty : \mathcal{M}(B) \to \mathcal{M}(B)$ defined by $\delta_\infty(a) := \sum s_n a s_n^*$ is a stricty continuous endomorphism of $\mathcal{M}(B)$ that is up to unitary equivalence uniquely defined by an arbitrary sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ with $\sum s_n s_n^*$, because $u := \sum t_n s_n^*$ is a unitary in $\mathcal{M}(B)$ if $\sum_n t_n t_n^* = 1 = t_n^* t_n$.

The commutant $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ contains a copy $C^*(s, t)$ of the Cuntz algebra $\mathcal{O}_2$ unitally.

A generalized W.-vN. Theorem:

Let B a σ-unital C^*-algebra, C a separable C^*-subalgebra of $\mathcal{M}(B)$, and $V\colon C \to \mathcal{M}(B)$ is a completely positive contraction that satisfies the following conditions (α) and (β):

(α) There exists $h \in C_+$ with $V(h) = 0$ and $h^{1/n}d \to d$ if $n \to \infty$ for every $d \in B$.

(β) For every $a \in B_+$, every finite subset $X \subset C_+$ and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$ with $\|d^*cd - aV(c)a\| < \varepsilon$ for $c \in X$.

One can show that the element $d \in \mathcal{M}(B)$ in (β) can be chosen such that $\|d\| \leq \|a\|$. There are contractions $e_n \in C^*(e)_+$, for a strictly positive contraction $e \in B$, such that (e_n) is an approximate unit of B that commutes sufficiently fast with the elements of a filtration $X_1 \subset X_2 \subset \cdots$ of C by finite-dimensional subspaces $X_n \subset C$, and satisfies $e_{n+1} e_n = e_n$. Find a sequence h_n of contractions in $C^*(h)_+$ such that $h_{n+1} h_n = h_n$ and that $\max_{m \leq n} \|e_m - h_n e_m\|$ sufficiently fast converges to zero. Build elements $t_n \in B$ from d_n with $\sup_{x \in X_n} \|d_n^*(1 - h_{n+2}) x (1 - h_{n+2}) d_n - e_{n+3} x e_{n+3}\|$ converging to zero as products of the form $t_n := (h_{n+1} - h_n) d_n e_{n+2}$. Then a suitable selection leads to strictly convergent series of the kind $S_m := t_{2m} e_m^{1/2} + \sum_{n>m} t_{2n} (e_n - e_{n-1})^{1/2}$, that satisfy part (i) of the following theorem. The other parts follow by passage to $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B)) / \mathrm{C_0}(\mathbb{R}_+, B)$.

Proposition (Generalized W.-vN. Theorem)

The above assumptions (α) and (β) imply:

(i) *There exist contractions $S_n \in \mathcal{M}(B)$ such that $S_n^* c S_n - V(c) \in B$ and $\lim_n \| S_n^* c S_n - V(c) \| = 0$ for $c \in C$ and $k > 0$.*

(ii) *If B is stable and $T \colon C \to \mathcal{M}(B)$ is a completely positive contraction such that $V := \delta_\infty \circ T$ satisfies (α) and (β), then there is a norm-continuous map $t \in \mathbb{R}_+ \mapsto S(t) \in \mathcal{M}(B)$ into the contractions of $\mathcal{M}(B)$ such that $S(t)^* c S(t) - T(c) \in B$ for $t \geq 0$ and $\lim_{t\to\infty} \| T(c) - S(t)^* c S(t) \| = 0$ for all $c \in C$.*

(iii) *If, in addition to (ii), T is a *-homomorphism and $T(C)B$ is dense in B, then id_C asymptotically absorbs T, i.e., $\mathrm{id}_C \oplus T \colon C \to \mathcal{M}(B)$ and id_C are unitarily homotopic (cf. following Definition).*

(iv) *The S_n in (i) (resp. $S(t)$ in (ii)) can be chosen as isometries if $1_{\mathcal{M}(B)} \in C$ and V (resp. T in (ii)) is unital.*

Definition (Unitary homotopy)

Let $h_j\colon A \to \mathcal{M}(B)$, $j = 1, 2$, *-homomorphisms, and let V a completely positive contraction from A into the multiplier algebra $\mathcal{M}(B)$ of B.

We call h_1 and h_2 **unitarily homotopic** if there is a *norm-continuous* map $t \mapsto U(t)$ from the non-negative real numbers $\mathbb{R}_+$ into the *unitaries* in $\mathcal{M}(B)$, such that,

(i) $U(t)^* h_1(a) U(t) - h_2(a) \in B$ for $t \in \mathbb{R}_+$ and $a \in A$, and

(ii) $\lim_{t\to\infty} \|h_2(a) - U(t)^* h_1(a) U(t)\|$ for all $a \in A$.

If $\mathcal{M}(B)$ contains a copy of $C^*(s,t)$ of $\mathcal{O}_2$ unitally, then we can define Cuntz addition $h_1 \oplus h_2 := sh_1(\cdot)s^* + th_2(\cdot)t^*$ on $\mathrm{Hom}(A, \mathcal{M}(B))$, and then (by definition) h_1 **asymptotically absorbs** h_2 if h_1 and $h_1 \oplus h_2$ are unitarily homotopic. Unitary *equivalence* implies unitary homotopy, but not necessarily point-norm homotopy of h_1 and h_2.

We say that h_1 **asymptotically dominates** a completely positive contraction $V\colon A \to \mathcal{M}(B)$ if there is a *norm-continuous* map $t \mapsto S(t)$ from the non-negative real numbers $\mathbb{R}_+$ into the *isometries* in $\mathcal{M}(B)$, such that,

(i) $S(t)^* h_1(a) S(t) - V(a) \in B$ for $t \in \mathbb{R}_+$ and $a \in A$, and
(ii) $\lim_{t\to\infty} \| V(a) - S(t)^* h_1(a) S(t) \|$ for all $a \in A$.

Note that h_1 and h_2 are unitarily homotopic (respectively h_1 asymptotically dominates V), if and only if, h_1 and h_2 are unitarily equivalent (respectively h_1 dominates V) in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B)) / \mathrm{C_0}(\mathbb{R}_+, B) \supset \mathcal{M}(B)$.

This implies e.g. that h_1 asymptotically absorbs h_2 if $h_2 \oplus h_2$ is unitarily homotopic to h_2 and h_1 asymptotically dominates $h_1 \oplus h_2$ in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B)) / \mathrm{C_0}(\mathbb{R}_+, B) \supset \mathcal{M}(B)$ - an algebraic calculation.

A unitary homotopy implies homotopy – in Hom$(A, \mathcal{M}(B))$ with point-norm topology – if the unitary group $\mathcal{U}(\mathcal{M}(B))$ of $\mathcal{M}(B)$ is connected in norm-topology (e.g. if B is stable).

Thus $h_1 \otimes \mathrm{id}_{\mathbb{K}}$ and $h_2 \otimes \mathrm{id}_{\mathbb{K}}$ are homotopic in Hom$(A \otimes \mathbb{K}, J \otimes \mathbb{K})$, if h_1 and h_2 are unitarily homotopic and if J is an *ideal* of $\mathcal{M}(B)$ that contains $h_1(A)$.

It is usual not always possible to find the map $t \mapsto U(t)$ with $U(0) = 1$ in the definition of unitary homotopy.

Let $\mathcal{C} \subset \mathrm{CP}(A, B)$ a non-degenerate countably generated m.o.c. cone, where A is separable, B is σ-unital, and both of A and B are stable.

There exists a *-homomorphism $H_0 \colon A \to \mathcal{M}(B)$ with following properties

(i) H_0 is faithful non-degenerate,

(ii) $\delta_\infty \circ H_0$ is unitarily equivalent to H_0,

(iii) $V_b := b^* H_0(\cdot) b \in \mathcal{C}$ for each $b \in B$, and

(iv) each $V \in \mathcal{C}$ can be approximated in point-norm topology by a sequence (V_{b_n}).

Corollary (Uniqueness of the universal bi-module for $\mathcal{C}$)

*If $H_1 \colon A \to \mathcal{M}(B)$ is a *-homomorphism that satisfies the conditions (i)–(iv) (in place of H_0), then H_1 and H_0 are unitarily homotopic. Each countably generated $\mathcal{C}$-compatible Hilbert A-B-bimodule (E, ϕ) is isomorphic to a sub-module of (B, H_0).*

The Hilbert B-module $\mathcal{H}_B \cong B$ is a "universal" Hilbert B-module (with trivial grading) and $H_0 \colon A \to \mathcal{L}(\mathcal{H}_B)$ the "universal" *-monomorphism that corresponds to the given cone $\mathcal{C}$.

Definition of regular subalgebras:

Let $C \subset A$ a C^*-subalgebra. C is regular for A if

(i) C separates the ideals J of A: $J_1 \cap C = J_2 \cap C$ implies $J_1 = J_2$.

(ii) $C \cap (J_1 + J_2) = (C \cap J_1) + (C \cap J_2)$ for all $J_1, J_2 \in \mathcal{I}(A)$.

Theorem (Realization of Ψ, H.H.,E.K.)

Suppose that B is separable and stable. Let $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ a non-degenerate lower s.c. action of $\mathrm{Prim}(B)$ on A.

*If $B \otimes \mathcal{O}_2$ contains a regular abelian C^*-subalgebra C then $\Psi = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathcal{C}_\Psi$. In particular, Ψ comes from a non-degenerate *-monomorphism $h\colon A \otimes \mathbb{K} \to \mathcal{M}(B)$, that is unique up to unitary homotopy of its infinite repeats.*

Corollary (Reconstruction Theorem, H.H.,E.K.)

Suppose that A is a nuclear and stable, that Ω is a sup–inf *closed sub-lattice of* $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ *with* $\mathrm{Prim}(A), \emptyset \in \Omega$ *. Then there is a non-degenerate *-monomorphism* $H_0 \colon A \to \mathcal{M}(A)$ *with following properties:*

(i) *The infinite repeat* $\delta_\infty \circ H_0$ *is unitarily equivalent to* H_0*.*

(ii) *For every* $U \in \mathbb{O}(\mathrm{Prim}(A))$ *holds* $H_0(J(V)) = H_0(A) \cap \mathcal{M}(A, J(U))$ *where* $V \in \Omega$ *is given by* $V = \bigcup\{W \in \Omega\, ;\; W \subset U\}$*.*

The H_0 *is uniquely determined up to unitary homotopy.*

Corollary (Continuation of Reconstruction Theorem)

The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ *of the Hilbert A-A-module* $\mathcal{H} := (A, H_0)$ *is stable and strongly purely infinite; and it is the same as the C*-Fock algebra* $\mathcal{F}(\mathcal{H})$ *of* $\mathcal{H}$*.*

The natural embedding of A into $\mathcal{O}_{\mathcal{H}}$ *defines a lattice isomorphism from* Ω *onto* $\mathbb{O}(\mathrm{Prim}(\mathcal{O}_{\mathcal{H}}))$ *and is a* $\mathrm{KK}(X; \cdot, \cdot)$*-equivalence for* $X := \mathrm{prime}(\Omega) \cong \mathrm{Prim}(\mathcal{O}_{\mathcal{H}})$*.*

Theorem (On Prim(A), H.H.,E.K.,M.Rørdam)

Let X a point-complete T_0-space. TFAE:

(i) $X \cong \text{Prim}(E)$ *for some exact C*-algebra* E.

(ii) *The lattice of open sets* $\mathbb{O}(X)$ *is isomorphic to an* sup–inf *invariant sub-lattice of* $\mathbb{O}(Y)$ *for some l.c. Polish space* Y.

(iii) *There is an l.c. Polish space* Y *and a pseudo-open and pseudo-surjective continuous map* $\pi\colon Y \to X$.

Theorem (On Prim(A), continuation)

If X satisfies (i)–(iii), then there is a stable nuclear C-algebra A with $A \cong A \otimes \mathcal{O}_2$, and a homeomorphism $\psi \colon X \to \mathrm{Prim}(A)$, such that,*

for every nuclear stable B with $B \otimes \mathcal{O}_2 \cong B$ and every homeomorphism $\phi \colon X \to \mathrm{Prim}(B)$,

there is an isomorphism $\alpha \colon A \to B$ with $\alpha(\psi(x)) = \phi(x)$ for $x \in X$.

The α is unique up to unitary homotopy.