Critical metrics on connected sums of Einstein four-manifolds

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Einstein manifolds

Einstein-Hilbert functional in dimension 4:

\[ \tilde{R}(g) = Vol(g)^{-1/2} \int_M R_g dV_g, \]

where \( R_g \) is the scalar curvature.
Einstein-Hilbert functional in dimension 4:

$$\tilde{R}(g) = \text{Vol}(g)^{-1/2} \int_M R_g dV_g,$$

where $R_g$ is the scalar curvature.

Euler-Lagrange equations:

$$\text{Ric}(g) = \lambda \cdot g,$$

where $\lambda$ is a constant.

$(M, g)$ is called an *Einstein manifold*. 
Orbifold Limits

Theorem (Anderson, Bando-Kasue-Nakajima, Tian)

\((M_i, g_i)\) sequence of 4-dimensional Einstein manifolds satisfying

\[
\int |Rm|^2 < \Lambda, \text{ diam}(g_i) < D, \text{ Vol}(g_i) > V > 0.
\]

Then for a subsequence \(\{j\} \subset \{i\},\)

\((M_j, g_j) \xrightarrow{\text{Cheeger–Gromov}} (M_\infty, g_\infty),\)

where \((M_\infty, g_\infty)\) is an orbifold with finitely many singular points.
Rescaling such a sequence to have bounded curvature near a singular point yields Ricci-flat non-compact limits called \textit{asymptotically locally Euclidean} spaces (ALE spaces), also called “bubbles”.

Example

There exists a sequence of Ricci-flat metrics $g_i$ on $K3$ satisfying:

$$(K3, g_i) \longrightarrow (T^4/\{\pm 1\}, g_{\text{flat}}).$$

At each of the 16 singular points, an Eguchi-Hanson metric on $T^*S^2$ “bubbles off”.

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In general, answer is “no”.

Question

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In general, answer is “no”.

Reason: this is a self-adjoint gluing problem so possibility of moduli is an obstruction.
Self-dual or anti-self-dual metrics

\((M^4, g)\) oriented.

\[
\mathcal{R} = \begin{pmatrix}
W^+ + \frac{R}{12} I & E \\
E & W^- + \frac{R}{12} I
\end{pmatrix}.
\]

\(E = Ric - (R/4)g.\)
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\[W^+ = 0\] is called anti-self-dual (ASD).

\[W^- = 0\] is called self-dual (SD).
Self-dual or anti-self-dual metrics

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\(W^- = 0\) is called self-dual (SD).

Either condition is conformally invariant.
Theorem (Donaldson-Friedman, Floer, Kovalev-Singer, etc.)

If $(M_1, g_1)$ and $(M_2, g_2)$ are ASD and $H^2(M_i, g_i) = \{0\}$ then there exist ASD metrics on the connected sum $M_1 \# M_2$. 
Theorem (Donaldson-Friedman, Floer, Kovalev-Singer, etc.)

If \((M_1, g_1)\) and \((M_2, g_2)\) are ASD and \(H^2(M_i, g_i) = \{0\}\) then there exist ASD metrics on the connected sum \(M_1 \# M_2\).

Contrast with Einstein gluing problem:

- ASD situation can be unobstructed \((H^2 = 0)\), yet still have moduli \((H^1 \neq 0)\).
- Cannot happen for a self-adjoint gluing problem.
Recently, Biquard showed the following:

**Theorem (Biquard, 2011)**

Let \((M, g)\) be a (non-compact) Poincaré-Einstein (P-E) metric with a \(\mathbb{Z}/2\mathbb{Z}\) orbifold singularity at \(p \in M\). If \((M, g)\) is rigid, then the singularity can be resolved to a P-E Einstein metric by gluing on an Eguchi-Hanson metric if and only if

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Self-adjointness of this gluing problem is overcome by freedom of choosing the boundary conformal class of the P-E metric.
A basis for the space of quadratic curvature functionals is

\[ \mathcal{W} = \int |W|^2 \, dV, \quad \rho = \int |Ric|^2 \, dV, \quad S = \int R^2 \, dV. \]
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In dimension four, the Chern-Gauss-Bonnet formula

\[ 32\pi^2 \chi(M) = \int |W|^2 \, dV - 2 \int |Ric|^2 \, dV + \frac{2}{3} \int R^2 \, dV \]

implies that \( \rho \) can be written as a linear combination of the other two (plus a topological term).
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Consequently, we will be interested in the functional

\[ \mathcal{B}_t[g] = \int |W|^2 \, dV + t \int R^2 \, dV. \]
Generalization of the Einstein condition

The Euler-Lagrange equations of $B_t$ are given by

$$B^t \equiv B + tC = 0,$$

where $B$ is the Bach tensor defined by

$$B_{ij} \equiv -4 \left( \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} \right),$$

and $C$ is the tensor defined by

$$C_{ij} = 2 \nabla_i \nabla_j R - 2(\Delta R) g_{ij} - 2RR_{ij} + \frac{1}{2} R^2 g_{ij}.$$
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- Any Einstein metric is critical for $B_t$.
- We will refer to such a critical metric as a $B^t$-flat metric.
For $t \neq 0$, by taking a trace of the E-L equations:

$$\Delta R = 0.$$ 

If $M$ is compact, this implies $R = \text{constant}$. 

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Consequently, the $B^t$-flat condition is equivalent to

$$B = 2tR \cdot E,$$

where $E$ denotes the traceless Ricci tensor.
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- The Bach tensor is a constant multiple of the traceless Ricci tensor.
The $B^t$-flat equation can be rewritten as

$$\Delta Ric = Rm \ast Rc.$$  \hspace{1cm} (*)

**Theorem (Tian-V)**

$(M_i, g_i)$ sequence of 4-dimensional manifolds satisfying (*) and

$$\int |Rm|^2 < \Lambda, \text{ Vol}(B(q, s)) > V s^4, b_1(M_i) < B.$$

Then for a subsequence $\{j\} \subset \{i\},$

$$(M_j, g_j) \xrightarrow{\text{Cheeger–Gromov}} (M_\infty, g_\infty),$$

where $(M_\infty, g_\infty)$ is a multi-fold satisfying (*), with finitely many singular points.
Introduction
The gluing procedure
The building blocks
Remarks on the proof

Question

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Can you reverse this process? I.e., start with an critical orbifold, “glue on” critical bubbles at the singular points, and resolve to a smooth critical metric? Answer is still “no” in general, because this is also a self-adjoint gluing problem. Our main theorem: the answer is “YES” in certain cases.
Main theorem

Theorem (Gursky-V 2013)

A $B^t$-flat metric exists on the manifolds in the table for some $t$ near the indicated value of $t_0$.

Table: Simply-connected examples with one bubble

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The constant $m_1$ is a geometric invariant called the mass of an certain asymptotically flat metric: the Green’s function metric of the product metric $S^2 \times S^2$. 
• $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ admits an $U(2)$-invariant Einstein metric called the Page metric. Does not admit any Kähler-Einstein metric, but the Page metric is conformal to an extremal Kähler metric.
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• $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ admits a toric invariant Einstein metric called the Chen-LeBrun-Weber metric. Again, does not admit any Kähler-Einstein metric, but the Chen-LeBrun-Weber metric is conformal to an extremal Kähler metric.
Remarks

- \( \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \) admits an \( U(2) \)-invariant Einstein metric called the Page metric. Does not admit any Kähler-Einstein metric, but the Page metric is conformal to an extremal Kähler metric.

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- \( S^2 \times S^2 \# S^2 \times S^2 \) does not admit any Kähler metric, it does not even admit an almost complex structure. Our metric is the first known example of a “canonical” metric on this manifold.
Green’s function metric

The conformal Laplacian:

$$L u = -6 \Delta u + R u.$$ 

If $\left( M, g \right)$ is compact and $R > 0$, then for any $p \in M$, there is a unique positive solution to the equation

$$LG = 0 \text{ on } M \setminus \{p\}$$

$$G = \rho^{-2} (1 + o(1))$$

as $\rho \to 0$, where $\rho$ is geodesic distance to the basepoint $p$. 
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- Denote $N = M \setminus \{p\}$ with metric $g_N = G^2g_M$. The metric $g_N$ is scalar-flat and asymptotically flat of order 2.
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- Denote \(N = M \setminus \{p\}\) with metric \(g_N = G^2 g_M\). The metric \(g_N\) is scalar-flat and asymptotically flat of order 2.
- If \((M, g)\) is Bach-flat, then \((N, g_N)\) is also Bach-flat (from conformal invariance) and scalar-flat (since we used the Green’s function). Consequently, \(g_N\) is \(B^t\)-flat for all \(t \in \mathbb{R}\).
Let \((Z, g_Z)\) and \((Y, g_Y)\) be Einstein manifolds, and assume that \(g_Y\) has positive scalar curvature.
The approximate metric

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- Convert \( (Y, g_Y) \) into an asymptotically flat (AF) metric \( (N, g_N) \) using the Green’s function for the conformal Laplacian based at \( y_0 \). As pointed out above, \( g_N \) is \( B^t \)-flat for any \( t \).
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- Let \(a > 0\) be small, and consider \(Z \setminus B(z_0, a)\). Scale the compact metric to \((Z, \tilde{g} = a^{-4}g_Z)\). Attach this metric to the metric \((N \setminus B(a^{-1}), g_N)\) using cutoff functions near the boundary, to obtain a smooth metric on the connect sum \(Z \# Y\).
The approximate metric

Since both $g_Z$ and $g_N$ are $B^t$-flat, this metric is an “approximate” $B^t$-flat metric, with vanishing $B^t$ tensor away from the “damage zone”, where cutoff functions were used.
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Total of 15 gluing parameters.
Lyapunov-Schmidt reduction

These 15 gluing parameters yield a 15-dimensional space of “approximate” kernel of the linearized operator. Using a Lyapunov-Schmidt reduction argument, one can reduce the problem to that of finding a zero of the *Kuranishi map* \( \Psi : U \subset \mathbb{R}^{15} \to \mathbb{R}^{15} \).
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- It is crucial to use certain weighted norms to find a bounded right inverse for the linearized operator.
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\[ \Psi : U \subset \mathbb{R}^{15} \rightarrow \mathbb{R}^{15}. \]

- It is crucial to use certain weighted norms to find a bounded right inverse for the linearized operator.
- This 15-dimensional problem is too difficult in general: we will take advantage of various symmetries in order to reduce to only 1 free parameter: the scaling parameter \( a \).
Technical theorem

The leading term of the Kuranishi map corresponding to the scaling parameter is given by:

**Theorem (Gursky-V 2013)**

As $a \to 0$, then for any $\epsilon > 0$,

$$
\Psi_1 = \left( \frac{2}{3} W(z_0) \otimes W(y_0) + 4t R(z_0) \text{mass}(g_N) \right) \omega_3 a^4 + O(a^{6-\epsilon}),
$$

where $\omega_3 = Vol(S^3)$, and the product of the Weyl tensors is given by

$$
W(z_0) \otimes W(y_0) = \sum_{ijkl} W_{ijkl}(z_0) (W_{ijkl}(y_0) + W_{ilkj}(y_0)),
$$

where $W_{ijkl}(\cdot)$ denotes the components of the Weyl tensor in a normal coordinate system at the corresponding point.
The Fubini-Study metric

$$(\mathbb{CP}^2, g_{FS})$$, the Fubini-Study metric, $Ric = 6g.$
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Torus action:

$$[z_0, z_1, z_2] \mapsto [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2].$$
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Flip symmetry:

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\]
The Fubini-Study metric

Figure: Orbit space of the torus action on $\mathbb{CP}^2$. 
The product metric

\((S^2 \times S^2, g_{S^2 \times S^2})\), the product of 2-dimensional spheres of Gaussian curvature 1, \(Ric = g\).
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Torus action:

Product of rotations fixing north and south poles.
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Flip symmetry:

\((p_1, p_2) \mapsto (p_2, p_1)\).
The product metric

Figure: Orbit space of the torus action on $S^2 \times S^2$. 
Recall the mass of an AF space is defined by

$$\text{mass}(g_N) = \lim_{R \to \infty} \frac{1}{\omega_3^3} \int_{S(R)} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii})(\partial_i \downarrow dV),$$

with $\omega_3 = \text{Vol}(S^3)$. 
Recall the mass of an AF space is defined by

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with $\omega_3 = Vol(S^3)$.

The Green’s function metric of the Fubini-Study metric $\hat{g}_{FS}$ is also known as the Burns metric, and is completely explicit, with mass given by

$$\text{mass}(\hat{g}_{FS}) = 2.$$
However, the Green’s function metric $\hat{g}_{S^2 \times S^2}$ of the product metric does not seem to have a known explicit description. We will denote

$$m_1 = \text{mass}(\hat{g}_{S^2 \times S^2}).$$

By the positive mass theorem of Schoen-Yau, $m_1 > 0$. Note that since $S^2 \times S^2$ is spin, this also follows from Witten’s proof of the positive mass theorem.
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**Remark**

*For the curious, the mass $m_1 = .5872...$, which implies that\(-9m_1)^{-1} = -.1892.....*
• We impose the toric symmetry and “flip” symmetry in order to reduce the number of free parameters to 1 (only the scaling parameter). That is, we perform an equivariant gluing.
Remarks on the proof

- We impose the toric symmetry and “flip” symmetry in order to reduce the number of free parameters to 1 (only the scaling parameter). That is, we perform an equivariant gluing.
- The special value of $t_0$ is computed by

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\]
• This choice of $t_0$ makes the leading term of the Kuranishi map vanish, and is furthermore a nondegenerate zero (if $R(z_0) > 0$; $\text{mass}(g_N) > 0$ by the positive mass theorem).
First case

Table: Simply-connected examples with one bubble

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- The compact metric is the Fubini-Study metric, with a Burns AF metric glued on, a computation yields $t_0 = -1/3$. 
Second case

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- The compact metric is the product metric on $S^2 \times S^2$, with a Burns AF metric glued on, this gives $t_0 = -1/3$.
- Alternatively, take the compact metric to be $(\mathbb{CP}^2, g_{FS})$, with a Green’s function $S^2 \times S^2$ metric glued on. In this case, $t_0 = -(9m_1)^{-1}$. 
The compact metric is the product metric on $S^2 \times S^2$, with a Green’s function $S^2 \times S^2$ metric glued on. In this case, $t_0 = -2(9m_1)^{-1}$.
Other symmetries

By imposing other symmetries, we can perform the gluing operation with more than one bubble:

**Table:** Simply-connected examples with several bubbles

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<td>$3# S^2 \times S^2$</td>
<td>$-2(9m_1)^{-1}$</td>
<td>bilateral</td>
</tr>
<tr>
<td>$S^2 \times S^2 # 2\overline{CP^2} = CP^2 # 3\overline{CP^2}$</td>
<td>$-1/3$</td>
<td>bilateral</td>
</tr>
<tr>
<td>$CP^2 # 3\overline{CP^2}$</td>
<td>$-1/3$</td>
<td>trilateral</td>
</tr>
<tr>
<td>$CP^2 # 3(S^2 \times S^2) = 4CP^2 # 3\overline{CP^2}$</td>
<td>$-(9m_1)^{-1}$</td>
<td>trilateral</td>
</tr>
<tr>
<td>$S^2 \times S^2 # 4\overline{CP^2} = CP^2 # 5\overline{CP^2}$</td>
<td>$-1/3$</td>
<td>quadrilateral</td>
</tr>
<tr>
<td>$5# S^2 \times S^2$</td>
<td>$-2(9m_1)^{-1}$</td>
<td>quadrilateral</td>
</tr>
</tbody>
</table>
We can also use the quotient spaces $G(2, 4)$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$ as building blocks to obtain non-simply connected examples. We do not list all of the examples here, but just note that we find a critical metric on $G(2, 4) \# G(2, 4)$, which has infinite fundamental group, and therefore does not admit any positive Einstein metric by Myers’ Theorem.
Technical Points

- **Ellipticity and gauging.** The $B^t$-flat equations are not elliptic due to diffeomorphism invariance. A gauging procedure analogous to the Bianchi gauge is used.
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- **Rigidity of $g_{FS}$ and $g_{S^2 \times S^2}$.** Proved recently by Gursky-V (to appear in Crelle’s Journal). Extends earlier work of O. Kobayashi for the Bach tensor, and N. Koiso for the Einstein equations.
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• **Refined approximate metric.** The approximate metric described above is not good enough. Can be improved by matching up leading terms of the metrics by solving certain auxiliary linear equations, so that the cutoff function disappears from the leading term. This step is inspired by the work of O. Biquard mentioned above.
Ellipticity and gauging

The linearized operator of the $B^t$-flat equation is not elliptic, due to diffeomorphism invariance. However, consider the “gauged” nonlinear map $P$ given by

$$P_g(\theta) = (B + tC)(g + \theta) + \mathcal{K}_{g+\theta}[\delta_g \mathcal{K}_g \delta_g \theta],$$

where $\mathcal{K}_g$ denotes the conformal Killing operator,

$$(\mathcal{K}_g \omega)_{ij} = \nabla_i \omega_j + \nabla_j \omega_i - \frac{1}{2}(\delta_g \omega) g_{ij},$$

$\delta$ denotes the divergence operator,

$$(\delta_g h)_j = \nabla^i h_{ij},$$

and

$$\circ \theta = \theta - \frac{1}{4} \text{tr}_g \theta g,$$

is the traceless part of $\theta$. 
Ellipticity and gauging

Let $S^t \equiv P'(0)$ denote the linearized operator at $\theta = 0$.

**Proposition**

If $t \neq 0$, then $S^t$ is elliptic. Furthermore, if $P(\theta) = 0$, and $\theta \in C^{4,\alpha}$ for some $0 < \alpha < 1$, then $B^t(g + \theta) = 0$ and $\theta \in C^\infty$. 
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- Proof is an integration-by-parts. Uses crucially that the $B^t$-flat equations are variational (recall $B_t$ is the functional), so $\delta B^t = 0$. Equivalent to diffeomorphism invariance of $B_t$. 
Rigidity

For $h$ transverse-traceless (TT), the linearized operator at an Einstein metric is given by

$$S^t h = \left( \Delta_L + \frac{1}{2} R \right) \left( \Delta_L + \left( \frac{1}{3} + t \right) R \right) h,$$

where $\Delta_L$ is the Lichnerowicz Laplacian, defined by

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ijpq} h^{pq} - \frac{1}{2} R h_{ij}.$$
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- N. Koiso previously studied infinitesimal Einstein deformations given by TT kernel of the operator $\Delta_L + \frac{1}{2} R$. 
Rigidity

For \( h = fg \), we have

\[
tr_g(S^t h) = 6t(3\Delta + R)(\Delta f). \tag{1}
\]

The rigidity question is then reduced to a separate analysis of the eigenvalues of \( \Delta_L \) on transverse-traceless tensors, and of \( \Delta \) on functions.
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Theorem (Gursky-V)

On $(\mathbb{C}P^{2}, g_{FS})$, $H_{t}^{1} = \{0\}$ provided that $t < 1$. 

Jeff Viaclovsky

Critical metrics on connected sums of Einstein four-manifolds
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**Theorem (Gursky-V)**

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**Theorem (Gursky-V)**

On \((S^2 \times S^2, g_{S^2 \times S^2})\), \( H^1_t = \{0\} \) provided that \( t < 2/3 \) and \( t \neq -1/3 \). If \( t = -1/3 \), then \( H^1_t \) is one-dimensional and spanned by the element \( g_1 - g_2 \).
Rigidity

- Positive mass theorem says that $t_0 < 0$, so luckily we are in the rigidity range of the factors.
Rigidity

- Positive mass theorem says that \( t_0 < 0 \), so luckily we are in the rigidity range of the factors.
- Gauge term is carefully chosen so that solutions of the linearized equation must be in the transverse-traceless gauge. That is, if \( S^t h = 0 \) then
  \[
  (B^t)'(h) + \mathcal{K} \delta \mathcal{K} \delta h = 0
  \]
  implies that separately,
  \[
  (B^t)'(h) = 0 \quad \text{and} \quad \delta h = 0.
  \]
Let $(Z, g_Z)$ be the compact metric. In Riemannian normal coordinates,

$$(g_Z)_{ij}(z) = \delta_{ij} - \frac{1}{3} R_{ikjl}(z_0) z^k z^l + O(|z|^4)_{ij}$$

as $z \to z_0$. 

Let $(N, g_N)$ be the Green's function metric of $(Y, g_Y)$, then we have

$$(g_N)_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(y_0) x^k x^l |x|^4 + 2 A_1 |x|^2 \delta_{ij} + O(|x|^{−4+\epsilon})$$
as $|x| \to \infty$, for any $\epsilon > 0$.

The constant $A$ is given by

$$\text{mass}(g_N) = 12 A - R(y_0) / 12.$$
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We consider \(a^{-4} g_Z\) and let \(z = a^2 x\), then we have

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- Linear equation on AF metric is obstructed, and this is how the leading term of the Kuranishi map is computed.
The obstruction

On \((N, g_N)\), solve

\[ Sh = 0 \]

\[ h = -a^4 \frac{1}{3} R_{ikjl}(z_0) x^k x^l + O(|x|^\epsilon), \]

as \(x \to \infty\).
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This equation is obstructed! Instead solve modified equation

\[ Sh = \lambda \cdot k_1, \]

where \(k_1\) pairs nontrivially with the decaying cokernel on the AF space \(o_1\).
The obstruction

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Similar procedure on compact piece, except this is unobstructed (compact piece is rigid).
Computation of leading term

\[ \lambda = \lim_{r \to \infty} \int_{B(r)} \langle Sh, o_1 \rangle \]

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This limit can be computed explicitly using the expansion

\[ (o_1)_{ij} = \frac{2}{3} W_{ikjl}(y_0) \frac{x^k x^l}{|x|^4} + f g_{ij} + O(|x|^{-4+\epsilon}) \]

as \( x \to \infty \), where \( f \) satisfies

\[ \Delta f = -\frac{1}{3} \langle Ric, o_1 \rangle, \]

and recalling that

\[ h = -a^4 \frac{1}{3} R_{ikjl}(z_0) x^k x^l + O(|x|^\epsilon), \]

as \( x \to \infty \).
The proof shows that there is a dichotomy.
Final remarks

The proof shows that there is a dichotomy. Either

- (i) there is a critical metric at exactly the critical $t_0$, in which case there would necessarily be a 1-dimensional moduli space of solutions for this fixed $t_0$,
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- (ii) for each value of the gluing parameter $a$ sufficiently small, there will be a critical metric for a corresponding value of $t_0 = t_0(a)$.

To determine which case actually happens, one must calculate the next term in the expansion of the Kuranishi map.
Bonus slide: the mass of $S^2 \times S^2$

$$m_1 = \frac{2}{3} - 12 \sum_{j,k=0}^{\infty} \frac{(2j + 1)(2k + 1)}{\lambda_{j,k}} \left( \sum_{p=0}^{j} \sum_{q=0}^{k} c_{j,k}^{p,q} \tilde{f}_{p,q} \right),$$

where

$$\lambda_{j,k} = j(j + 1) + k(k + 1) + \frac{2}{3},$$

$$c_{j,k}^{p,q} = (-1)^{p+q} \binom{j}{p} \binom{j + p}{p} \binom{k}{q} \binom{k + q}{q},$$

$$\tilde{f}_{p,q} = -\frac{1}{108} \frac{1}{p + q + 3} \left\{ 29 + 32(p + q) + 21(p^2 + q^2) + 6(p^3 + q^3) \right.$$  

$$+ 4(-1)^{p+1}(p + 1)(p + 2)(3p^2 + 9p + 10)(\log(2) - A(p + 2))$$  

$$+ 4(-1)^{q+1}(q + 1)(q + 2)(3q^2 + 9q + 10)(\log(2) - A(q + 2)) \left. \right\},$$

where

$$A(p) = \sum_{i=1}^{p} (-1)^{i-1} \frac{1}{i}.$$