Commutators in semicircular systems

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Random Variables

Given a Hilbert space space H with distinguished vector Ω satisfying:

$$\langle \Omega \,|\, \Omega \rangle = 1$$

A *random variable* is simply a self-adjoint operator X acting on H.

The *moments* of the random variable X are given by:

$$\tau(X^n) = \langle X^n \Omega \,|\, \Omega \rangle$$

The random variables which we are interested in are uniquely determined by their moments.

Semicircular random variable

Let H denote the set of all real functions $f : [-2, 2] \to [-2, 2]$ which satisfy:

$$\frac{1}{2\pi}\int_{-2}^{2} f(x)^2 \sqrt{4 - x^2}\, dx < \infty$$

H is a Hilbert space with inner product:

$$\langle f|g\rangle = \frac{1}{2\pi}\int_{-2}^{2} f(x)g(x)\sqrt{4 - x^2}\, dx$$

We shall chose as distinguished vector the function $f(x) = 1$. The operator "multiplication by x" is self-adjoint and thus constitues a random variable. Its moments are given by the *catalan numbers*:

$$\frac{1}{2\pi}\int_{-2}^{2} x^{2n}\sqrt{4 - x^2} dx = \frac{1}{n+1}\binom{2n}{n}$$

Chebyshev polynomials

Let us define:

$$P(x,z) = \frac{1}{1 - xz + z^2} = \sum_n P_n(x) z^n$$

One may check that:

$$\langle P_n(x), P_m(x) \rangle = \delta_{n,m}$$

The Chebysheff polynomials satisfy the following three term recurrence:

$$xP_n(x) = P_{n+1}(x) + P_{n-1}(x)$$

Fock space

Fix a finite dimensional inner product space H with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ Define the full Fock space of H to be the metric completion of:

$$T(H) = \mathbb{C}\Omega \oplus_{n\geq 1} H^{\otimes n}$$

For each i let a_i be the operator:

$$a_i[v] = a_i \otimes v$$

The adjoint operator a_i^* acts via:

$$a_i^*[\Omega] = 0$$
$$a_i^*[v_1 \otimes v_2 \otimes \cdots \otimes v_n] = \langle e_i | v_i \rangle v_2 \otimes \cdots \otimes v_n$$

The operators $A_i = a_i + a_i^*$ are self-adjoint. It is not hard to convince oneself that:

$$\left\langle A_i^{2n}\Omega | \Omega \right\rangle = \frac{1}{n+1}\binom{2n}{n}$$

Von Neumann algebra

Let $\mathfrak{A}$ denote the *Von Neumann algebra* generated by the $A_i = a_i + a_i^*$. We have:

$$P_{i_1}(A_{j_1})P_{i_2}(A_{j_2})\cdots P_{i_k}(A_{j_k})\Omega = e_{i_1}^{\otimes j_1} \otimes e_{i_2}^{\otimes j_2} \otimes \cdots \otimes e_{i_k}^{\otimes j_k}$$

For $j_1, j_2, \dots j_k \in \{1, 2, \dots n\}$ and $i_1, i_2, \dots, i_k \in \mathbb{N}_{>0}$ with $j_\ell \neq j_{\ell+1}$ for all $\ell \in \{1, \dots, k-1\}$.

The state τ is a *trace*, that is:

$$\tau(XY) = \tau(YX) \text{ for all } X, Y \in \mathfrak{A}$$

Commutators

We wish to show that for any $t \in \{1, 2, \dots n\}$, the commutant of X_t is the subalgebra generated by X_t.

$$\boxed{X_t' = \mathrm{alg}(X_t)}$$

Observe firstly that:

$$\tau(a[X_t, b]) = \tau(b[a, X_t])$$

This implies that elements of the form $[b, X_t]$ are orthogonal to the commutant of X_t.

$$\{[b, X_t], b \in \mathfrak{A}\} \subseteq (X_t')^\perp$$

We shall show that any element in the orthogonal complement of the commutant of $\mathrm{alg}(X_t)$ can be approximated by commutators of the form $[b, X_t]$

$$(\mathrm{alg}(X_t))^\perp \subseteq \overline{\{[b, X_t], b \in \mathfrak{A}\}} \subseteq (X_t')^\perp$$

Projection formula

Fix some $t \in \{1, 2, \ldots, n\}$. Let S be any element of the form:

$$P_{i_1}(X_{j_1})P_{i_2}(X_{j_2}) \cdots P_{i_k}(X_{j_k})$$

with $j_1 \neq t \neq i_k$. For each n, k let:

$$S_{n,k} = P_n(X_t)\, S\, P_k(X_t)$$

We have:

$$\boxed{S = \lim_{m \to \infty} \frac{1}{2m+3} \sum_{k=0}^{m-1} (m-k)\left([S_{k,k+1}, X_t] - [S_{k+1,k}, X_t]\right)}$$

Projection II

If a commutes with x then:

$$[x, ab] = a[x, b]$$

since multiplication is commutative, it follows that:

$$\begin{aligned} S_{n,k} &= \lim_{m\to\infty} \frac{1}{2m+3} \sum_{i=0}^{m-1} P_n(X_t) \left([S_{i,i+1}, X_t] - [S_{i+1,i}, X_t] \right) P_k(X_t) \\ &= \lim_{m\to\infty} \frac{1}{2m+3} \sum_{i=0}^{m-1} \left([P_n(X_t) S_{i,i+1} P_k(X_t), X_t] - [P_n(X_t) S_{i+1,i} P_k(X_t), X_t] \right) \end{aligned}$$

Conclusion

We have shown that for each $t \in \{1, \ldots, n\}$ we have:

$$\boxed{X_t' = \mathrm{alg}(X_t)}$$

In particular, this implies that the center of $\mathfrak{A}$ is trivial.