The limiting distributions of permutation invariant matrices

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Introduction: the spectrum of permutation invariant random matrices

Intro: the large permutation invariant random matrices

A family $\mathbf{A}_N = (A_j)_{j\in J}$ of N by N random matrices is called **permutation invariant** whenever

$$\mathbf{A}_N \stackrel{\mathcal{L}}{=} (UA_jU^*)_{j\in J}$$

for any permutation matrix U.

Theorem (Weak-asymptotic freeness of permutation matrices)

Let $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(p)}$ be independent and permutation invariant families of N by N matrices. Assuming a **moment** *and a* **decorrelation** *hypothesis on each family, we characterize the joint limiting *-distribution of $(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(p)})$*

The moment condition is the convergence of $\mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\, t(\mathbf{A}_N^{(j)})\big]$ for functionals t that generalize *-polynomials.

Interest:

1. Unified proof of the asymptotic *-freeness of Wigner, unitary invariant and deterministic matrices.
2. Characterize the limiting distribution of "heavy Wigner" and deterministic matrices.
3. Rich connections with two theories of convergence of graphs (sparse and dense graphs).
4. Based on the moments methods.
5. Can adapt the formalism depending on the problem to maximize the expressiveness/additional-structure ratio.

Limitations: cannot be an analytic theory, need combinatorics (related to Nica-Speicher obstruction of the existence of notions of independence/freeness for n.c.r.v.)

Task of the talk: to present

1. the structure that enriches *-probability spaces,
2. the associated notion of freeness,

In order to formulate the Theorem in terms of convergence towards free variables.

Technical aspects and proofs in two weeks.

A new notion of variables

Space of variables

A *-**probability space** is

1. a unital *-algebra $\mathcal{A}$,
2. endowed with a unital, tracial linear form Φ
3. that satisfies the positivity condition $\Phi(a^*a) \geq 0$.

We consider *-probability with more structure, where

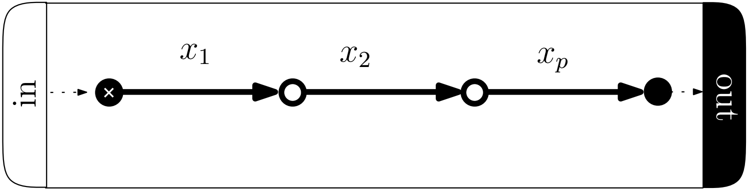
1. the space is an operad algebra over a space of new functionals the *-**graph polynomials**,
2. where Φ is written in terms of a functional τ,
3. that satisfies a positivity condition.

Equivalently, item 1 means that $\mathcal{A}$ is a *-Frobenius object (category-theoretical definition).

New operations on matrices

*-polynomials:

$$A_1 \times \cdots \times A_p(i,j) = \sum_{i_2,\dots,i_{p-1}=1}^{N} A_1(i,i_2)\dots A_p(i_{p-1},j).$$

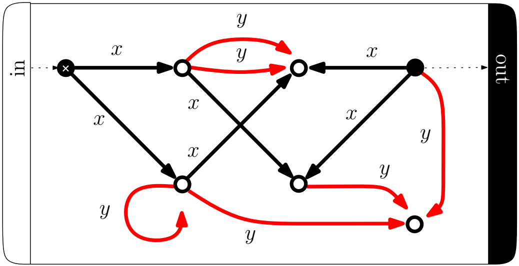


We generalize the linear composition as follow (following Mingo and Speicher)

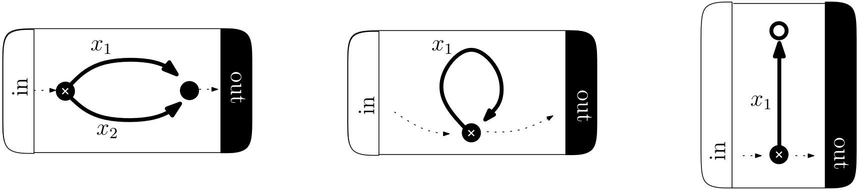
***-graph polynomials:** Let $\mathbf{A}_N = (A_j)_{j\in J}$ be a family of $N \times N$ matrices.
A *-graph monomial is the collection t of

1. A finite, connected graph (V, E)
2. a labeling of the edges by indeterminates $(x_j, x_j^*)_{j\in J}$
3. two marked vertices, the "input' and the "output'

We then set $t(\mathbf{A}_N) = \sum_{\substack{\phi: V\to\{1,\dots,N\} \\ s.t. \Phi(in)=i, \Phi(j)=out}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v),\phi(w)\big)$



Examples of operations:



$t(\mathbf{A}_N)(i,j) = A_1(i,j) \times A_2(i,j) \Rightarrow$ Hadamard (entry-wise) product.

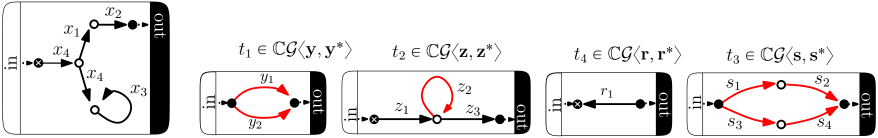
$t(\mathbf{A}_N)(i,j) = \delta_{i,j} A_1(i,i) \Rightarrow$ Projection on the diagonal.

$t(\mathbf{A}_N)(i,j) = \delta_{i,j} \sum_{k=1}^N A_1(i,k)$
$\Rightarrow \; t(\mathbf{A}_N) = diag\Big(\sum_{k=1}^N A_1(i,i) \Big)_{i=1,\dots,N} = deg(\mathbf{A}_N).$

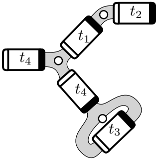
Structure of the space of *-graph polynomials

The space $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ of *-graph polynomials is an **operad**, i.e. one can replace the variables of a *-graph monomial by *-graph monomials and get a new *-graph monomial.

$t \in \mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ $\quad t_1 \in \mathbb{C}\mathcal{G}\langle \mathbf{y}, \mathbf{y}^* \rangle$ $\quad t_2 \in \mathbb{C}\mathcal{G}\langle \mathbf{z}, \mathbf{z}^* \rangle$ $\quad t_4 \in \mathbb{C}\mathcal{G}\langle \mathbf{r}, \mathbf{r}^* \rangle$ $\quad t_3 \in \mathbb{C}\mathcal{G}\langle \mathbf{s}, \mathbf{s}^* \rangle$



Substitution:



glueing the vertices

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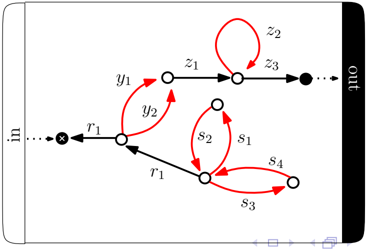


A **space of traffics** is a *-probability where one can replace the variables of a *-graph monomial by non commutative variables and get a new variable.

Examples:

1. The random matrices.
2. The random networks: given a possibly infinite set $\mathcal{V}$, $\mathbf{A}$ is a family of locally infinite matrices indexed in $\mathcal{V}^2$:

$$t(\mathbf{A})(v,w) = \sum_{\substack{\phi: V \to \mathcal{V} \\ \phi(\mathrm{in})=w,\ \phi(\mathrm{out})=v}} \prod_{e=(v',w')\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v'),\phi(w')\big)$$

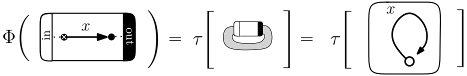
3. The random rooted graphs with locally finite degree, for which $t(\mathbf{A})(v,w)$ counts homomorphisms.
4. The random groups with given generators: for any *-graph monomial t, $\exists P, P_1, \dots, P_n$ such that for any group Γ with generators $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_p)$

$$t(\boldsymbol{\gamma}) = P(\boldsymbol{\gamma}) 1_{P_1(\boldsymbol{\gamma}) = \cdots = P_p(\boldsymbol{\gamma}) = e}.$$

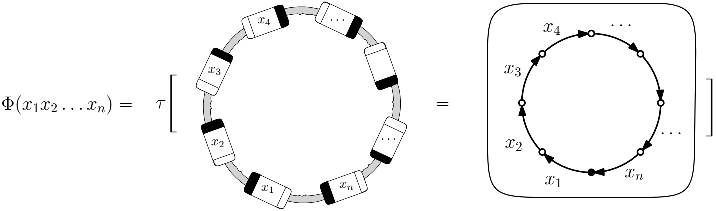
Traciality

The distribution of traffics of a family $\mathbf{a} = (a_j)_{j \in J}$ is the map $t \mapsto \Phi(t)$ defined on the space of *-graph polynomials.

Traciality: We assume that for *-graph monomials t, $\Phi(t)$ depends only on the graph obtained by merging the input and the output.



glueing the vertices

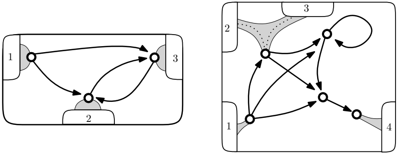


glueing the vertices

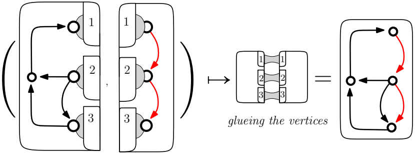
For random graphs, traciality is related to the notion of **unimodularity**.

The positivity condition

multi-rooted *-graph polynomials labelled graph with sequence of in/outputs.



Given two n-rooted *-graph monomials t_1 and t_2, one obtains a labelled graph $T(t_1, t_2)$ as follow:



We assume $\tau\big[T(t^*, t)\big] \geq 0$ for any n^*-graph polynomial t.

Application:

Proposition (Degenerated traffic variables)

Let a be a traffic variable in a space of traffics with traffic state τ and tracial state Φ. Then, the two following conditions are equivalent.

(1) *For any *-test graph T in one variable and at least one edge, one has $\tau\big[T(a)\big] = 0$,*

(2) $\Phi(a^*a) = \Phi\big(deg(a)^*deg(a)\big) = \Phi\big(deg(a^*)^*deg(a^*)\big) = 0.$

Let J_N be the matrix whose entries are $\frac{1}{N}$. It converges in distribution of traffics to a non trivial traffic-variable with null variance: for any *-test graph T in one variable, one has

$$\tau_N\big[T(J_N)\big] \underset{N\to\infty}{\longrightarrow} 1_{T \text{ is a tree}}.$$

Hence, J_N converges in distribution of traffics to a non trivial limit who has variance zero.

A new notion of freeness

An analogue of cumulants

Classical cumulants: linear maps $(\kappa_m^{(1)})_{m\geq 1}$ given by

$$\mathbb{E}(X_1 \dots X_n) = \sum_{\pi \in \mathcal{P}(n)} \prod_{B=\{i_1<\dots<i_m\}\in\pi} \kappa_m^{(1)}(X_{i_1}, \dots, X_{i_m}).$$

Free cumulants: linear maps $(\kappa_m^{(2)})_{m\geq 1}$ given by

$$\Phi(a_1 \dots a_n) = \sum_{\pi \in NC\mathcal{P}(n)} \prod_{B=\{i_1<\dots<i_m\}\in\pi} \kappa_m^{(2)}(a_{i_1}, \dots, a_{i_m}).$$

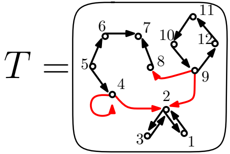
Analogue for traffic: linear map τ^0 defined on labelled graphs given by

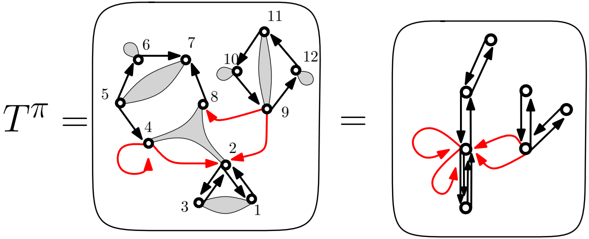
$$\tau\big[T(\mathbf{a})\big] = \sum_{\pi \in \mathcal{P}(V)} \tau^0\big[T^\pi(\mathbf{a})\big],$$

where T^π is obtained by merging the vertices of T that belong to a same block of π.

T test graph on $V = \{1, \dots, 12\}$

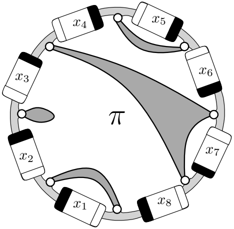
$\pi = \Big\{\{1,3\}, \{2,4,8\}, \{5,7\}, \{6\}, \{9,11\}, \{10\}, \{12\}\Big\}$





Free cumulants = partitions of edges
Traffic analogue = partitions of vertices

$\Phi(x_1x_2\dots x_n) = \sum_{\pi\in\mathcal{P}(n)} \tau^0 \Big[$



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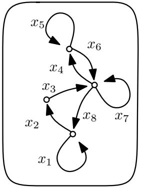
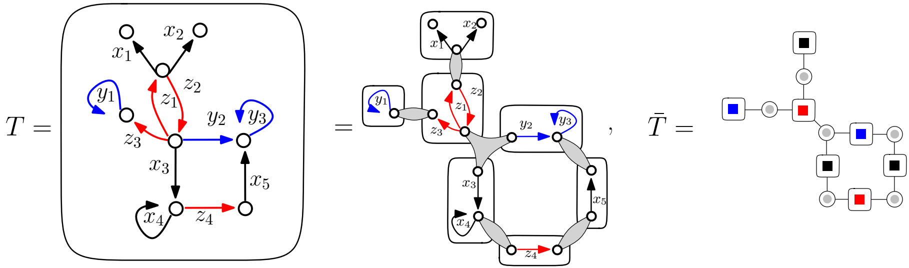


$\Big]$

Related to free cumulants by the Kreweras duality.

Free product of distributions of traffics

A labelled graph T in families of variables $\mathbf{x}_1, \dots, \mathbf{x}_p$ (with pairwise different indeterminates) is called a free product whenever the reduced graph $\bar{T}$ is a tree:



Definition (Traffic-freeness)

Families of traffic variables $\mathbf{a}_1, \dots, \mathbf{a}_p$ *are traffic free whenever: for any* T

$$\tau^0\big[T(\mathbf{a}_1, \dots, \mathbf{a}_p)\big] = \begin{cases} \prod_{\tilde T} \tau^0\big[\tilde T(\mathbf{a}_{i_{\tilde T}})\big] & \text{if } T \text{ free product in } \mathbf{x}_1, \dots, \mathbf{x}_p \\ 0 & \text{otherwise,} \end{cases}$$

where the product is over the colored connected components.

$\Rightarrow$ Families of independent and permutation invariant families of matrices (with a technical condition) that converge in distribution of traffics converge joint to traffic-free families of traffics.

Thank you for you attention