Structured Random Unitary Matrices and Asymptotic Freeness

Brendan Farrell

California Institute of Technology

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Joint work with Greg Anderson

Theorem (D.-V. Voiculescu, 1991)

Let $\mathcal{U}_N^{(1)}$ and $\mathcal{U}_N^{(2)}$ be independent, Haar-distributed unitary matrices of size $N \times N$ and $\{A_N\}_{N=1}^{\infty}$ and $\{B_N\}_{N=1}^{\infty}$ sequences of (nonrandom) uniformly bounded self-adjoint matrices of size $N \times N$ with spectral measures converging to μ_A and μ_B. Then, as $N \to \infty$,

$$\mathcal{U}_N^{(1)} A_N \mathcal{U}_N^{(1)*} \quad \text{and} \quad \mathcal{U}_N^{(2)} B_N \mathcal{U}_N^{(2)*}$$

are asymptotically free.

Gives a limit law for $\mathcal{U}_N^{(1)} A_N \mathcal{U}_N^{(1)*} + \mathcal{U}_N^{(2)} B_N \mathcal{U}_N^{(2)*}$

and

$$A_N \mathcal{U}_N^{(1)} B_N \mathcal{U}_N^{(1)*} A_N$$

in terms of μ_A and μ_B.

Simplest application:

$$A := P_2 \mathcal{U}_N P_1 \mathcal{U}_N^* P_2,$$

P_1 and P_2 are orthogonal projections with ranks pN and qN. Let

$$F(x) = \frac{1}{N} \sharp\{\lambda_i(A) \leq x\}.$$

Wachter (1980): when $F(x)$ converges almost surely to the distribution function with density

$$\begin{aligned} f(x) \quad := \quad & \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi x(1-x)} I_{[\lambda_-,\lambda_+]}(x) \\ & +(1 - \min(p, q))\delta_0(x) + (\max(p + q - 1, 0))\delta_1(x), \end{aligned}$$

where

$$\lambda_\pm := p + q - 2pq \pm \sqrt{4pq(1-p)(1-q)}.$$

Simplest example of multiplicative free convolution.

See Capitaine and Casalis (2004), B. Collins (2005).

$P, Q \in \mathbb{R}^{N\times N}$: random coordinate projections with independent diagonal entries:

- $P_{i,i} = 1$ with probability $(1 - p)$
- $Q_{i,i} = 1$ with probability $(1 - q)$.

$F \in \mathbb{C}^{N\times N}$: discrete Fourier transform matrix

$$F_{j,k} = \frac{1}{\sqrt{N}} e^{2\pi ijk/N}.$$

Theorem (B.F., 2011)

The empirical eigenvalue distribution of $PFQF^*P$ converges almost surely to f.

Same behavior as for $PUQU^*P$ where U has Haar distribution.

Suggests behavior related to freeness.

Definition

The sequence of sets of unitary matrices $\left\{\left\{U_N^{(i)}\right\}_{i\in I}\right\}_{N\in\mathbb{N}}$ is *asymptotically liberating* if for all $i_1,\ldots,i_\ell \in I$ satisfying

$$\ell \geq 2,\ \ i_1 \neq i_2,\ \ \ldots,\ \ i_{\ell-1} \neq i_\ell,\ \ i_\ell \neq i_1, \tag{1}$$

there exists $c(i_1,\ldots,i_\ell)$ such that

$$\left|\mathbb{E}\mathrm{tr}\left(U_{i_1}^{(N)} A_1 U_{i_1}^{(N)*} \cdots U_{i_\ell}^{(N)} A_\ell U_{i_\ell}^{(N)*}\right)\right| \leq c(i_1,\ldots,i_\ell)\|A_1\|\cdots\|A_\ell\|$$

for all constant matrices $A_1,\ldots,A_\ell \in \mathbb{C}^{N\times N}$ with trace zero.

- $\{U_i^{(N)}\}_{i\in I}$ set of random unitary matrices
- $\{\{T_{i,j}^{(N)}\}_{j\in J_i}\}_{i\in I}$ set of bounded self-adjoint matrices
- $\mathcal{A}_N$ algebra of $N\times N$ random matrices defined on the same space as $\{U_i^{(N)}\}_{i\in I}$.
- $\phi^{(N)}(A)=\frac{1}{N}\mathbb{E}\mathrm{tr}A$
- $\tau_i^{(N)}:\mathbb{C}\langle\{\mathbf{X}_{i,j}\}_{j\in J_i}\rangle\to\mathbb{C}$, the joint law of $\{T_{i,j}^{(N)}\}_{j\in J_i}$
- $\mu^{(N)}:\mathbb{C}\langle\{\{\mathbf{X}_{i,j}\}_{j\in J_i}\}_{i\in I}\rangle\to\mathbb{C}$, the joint law of

$$\{\{U_i^{(N)}T_{i,j}^{(N)}U_i^{(N)}\}_{j\in J_i}\}_{i\in I}$$

Lemma

Assume

- $\tau_i=\lim_{N\to\infty}\tau_i^{(N)}$ *exists for all* $i\in I$
- $\sup_N\max_{i\in I}\max_{j\in J_i}\|T_{i,j}^{(N)}\|<\infty$
- $\{\{U_{i,j}^{(N)}\}_{j\in J_i}\}_{i\in I}$ *is asymptotically liberating*

Then $\mu=\lim_{N\to\infty}\mu^{(N)}$ *exists and is tracial, and the rows of* $\{\{\mathbf{X}_{i,j}\}_{j\in J_i}\}_{i\in I}$ *are free from each other with respect to* μ.

$W \in \mathbb{R}^{N\times N}$ is a *random signed permutation matrix* if

$$W(i,j) = \epsilon_i \delta_{i,\sigma(j)},$$

where $\epsilon_1, \ldots, \epsilon_N \in \{\pm 1\}$ and $\sigma \in S_N$ is a permutation.

Theorem (G. Anderson and B. Farrell, 2013)

Let $\{U_{i\in I}^{(N)}\}$ be random unitary matrices. Assume:

- *For all N and deterministic signed permutation matrix W,* $$\left\{W^* U_{ii'}^{(N)} W\right\}_{\substack{i,i'\in I \\ \text{s.t. } i\neq i'}} \stackrel{d}{=} \left\{U_{ii'}^{(N)}\right\}_{\substack{i,i'\in I \\ \text{s.t. } i\neq i'}}. \tag{2}$$
- *For each positive integer ℓ* $$\sup_{N=1}^{\infty} \max_{\substack{i,i'\in I \\ \text{s.t. } i\neq i'}} \max_{\alpha,\beta=1}^{N} \sqrt{N}\left(\mathbb{E}\left|\left(U_i^{(N)*} U_{i'}^{(N)}\right)(\alpha,\beta)\right|\right)^{1/\ell} < \infty. \tag{3}$$

Then the sequence of families $\left\{\left\{U_i^{(N)}\right\}_{i\in I}\right\}_{N=1}^{\infty}$ is asymptotically liberating.

A matrix $H \in \mathbb{C}^{N\times N}$ is a *general Hadamard* matrix $\frac{1}{\sqrt{N}}H$ is unitary and $|H(i,j)| = 1$ for all $1 \le i,j \le N$.

Corollary

Assume:

- *I is a finite index set.*
- *$H^{(N)}$ is a general Hadamard matrix for each N.*
- *$W^{(N)}$ is uniformly distributed on signed permutation matrices.*
- *$\{D_i^{(N)}\}_{i\in I}$ are i.i.d., uniformly distributed signed permutation matrices, independent of $W^{(N)}$.*

Then the sequence

$$\left\{ \left\{ W^{(N)} \right\} \cup \left\{ \frac{H^{(N)}}{\sqrt{N}} W^{(N)} \right\} \cup \left\{ D_i^{(N)} \frac{H^{(N)}}{\sqrt{N}} W^{(N)} \right\}_{i\in I} \right\}_{N=1}^{\infty} \tag{4}$$

is asymptotically liberating.

Corollary

Assume:

- X *and* Y *are bounded real random variables with distributions* ν_X *and* ν_Y*.*
- $\{H^{(N)}\}_{N=1}^{\infty}$ *is a sequence of* N*-by-*N *Hadamard matrices.*
- $\{X^{(N)}\}_{N=1}^{\infty}$ *and* $\{Y^{(N)}\}_{N=1}^{\infty}$ *are independent sequences of* N*-by-*N *diagonal matrices with indep. copies of* X *and* Y*, respectively, on the diagonal.*
- $\mathcal{A}^{(N)}$ *is the algebra of random* N*-by-*N *matrices with essentially bounded complex entries defined on the same probability space as* $X^{(N)}$ *and* $Y^{(N)}$
- $\phi^{(N)}$ *is the state on* $\mathcal{A}^{(N)}$ *defined by* $\phi^{(N)}(A) = \mathbb{E}\frac{1}{N}\mathrm{tr}A$*.*

Then

$$X^{(N)} \quad \text{and} \quad \frac{1}{N}H^{(N)}Y^{(N)}H^{(N)*}$$

are asymptotically free.

From this we recover the earlier theorem.

$P, Q \in \mathbb{R}^{N\times N}$: random coordinate projections with independent diagonal entries:

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Two natural avenues to pursue from this point:

- Discrete uncertainty principles.
- Relationship to classical random matrix theory.

Relationship to uncertainty principles

U – unitary matrix
P_1 and P_2 – coordinate projections with support sets S_1 and S_2.

Suppose there exists x such that support$(x) \subset S_1$ and support$(Ux) \subset S_2$. Then

$$\|P_2 U P_1 x\|_2 = \|P_2 U x\|_2 = \|Ux\|_2 = \|x\|_2,$$

so that $\|P_2 U P_1\| = 1$.

If no such x exists, then $\|P_2 U P_1\| < 1$.

Thus, coordinate projections (very simple matrices) allow us to address an uncertainty principle.

This is also the simplest instance of free multiplicative convolution.

Classical random matrix theory

Ensemble	Matrix Form	Matrix Name	Law
Gaussian	$X = X^*$	Wigner	Semicircle law
Laguerre	XX^*	Sample covariance	Marchenko-Pastur Law
Jacobi	$P_2 \mathcal{U} P_1 \mathcal{U}^* P_2$	MANOVA	(Kesten law)

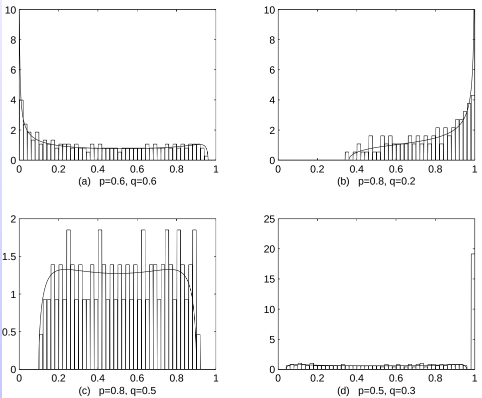

Figure: Plots for f_M for parameter pairs p, q