Exact separation phenomenon for the eigenvalues of large Information-Plus-Noise type matrices. Application to spiked models

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We consider large Information-Plus-Noise type matrices of the form $M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$ where X_N is an $n \times N$ ($n \leq N$) matrix consisting of independent standardized complex entries, A_N is an $n \times N$ nonrandom matrix and $\sigma > 0$. As N tends to infinity, if $c_N = n/N \to c \in]0, 1]$ and if the empirical spectral measure $\mu_{A_N A_N^*}$ of $A_N A_N^*$ converges weakly to some compactly supported probability distribution $\nu \neq \delta_0$, Dozier and Silverstein established that almost surely the empirical spectral measure of M_N converges weakly towards a nonrandom distribution $\mu_{\sigma,\nu,c}$. Bai and Silverstein proved, under certain assumptions on the model, that for some fixed closed interval in $]0; +\infty[$ outside the support of $\mu_{\sigma,\mu_{A_N A_N^*},c_N}$ for all large N, almost surely, no eigenvalues of M_N will appear in this interval for all N large. We show that there is an exact separation phenomenon between the spectrum of M_N and the spectrum of $A_N A_N^*$: to a gap in the spectrum of M_N pointed out by Bai and Silverstein, it corresponds a gap in the spectrum of $A_N A_N^*$ which splits the spectrum of $A_N A_N^*$ exactly as that of M_N. We deduce a relationship between the distribution functions of some probability measures on $\mathbb{R}^+$ and their rectangular free convolution with ratio c with the pushfoward of a Marchenko-Pastur distribution with parameter c by $x \mapsto \sqrt{x}$. We use the previous results to characterize the outliers of spiked Information-Plus-Noise type models.