New Examples of Noncommutative Brownian Motion

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(joint work with Marius Junge and Benoit Collins)

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Definition (Collins, Junge)

Let $\mathcal{M} = \vee_{t\geq 0}\mathcal{M}_t$ ($\mathcal{M}_s \subset \mathcal{M}_t$ for $s < t$) be a filtered finite von Neumann algebra. b_t is a Brownian motion if

1. b_t *is self-adjoint.*
2. $b_t \in \cap_{1\leq p<\infty} L^p(\mathcal{M},\tau) := \mathcal{M}^\infty$. *(where* $\|x\|_p := \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$*.)*
3. $t \to b_t$ *is continuous in* $\mathcal{M}^\infty$ *with respect to the natural topology induced from the* L^p*-norms.*
4. b_t *and* $b_t^2 - t$ *are martingales.* $(E_s(b_t) = b_s$ *for* $s < t$.)
5. $\|b_t - b_s\|_4 \leq C|t-s|^{\frac{1}{2}}$.
6. *Let* (I_k) *be a collection of disjoint intervals such that* $|I_k| = |I_j|$ *for all* k, j*. Let* $b_{I_k} := b_{s_k} - b_{r_k}$ *where* $I_k = [r_k, s_k)$*. The* (b_{I_k}) *are exchangeable, i.e. the sequences* $(b_{I_1},\ldots,b_{I_n})$ *and* $(b_{I_{\sigma(1)}},\ldots,b_{I_{\sigma(n)}})$ *are equal in distribution.*

Deformed Fock Spaces

Let $\varphi : S_\infty \to \mathbb{R}$ be a positive definite function, and H be a real Hilbert space.
Define $F_\varphi(H) = \oplus_{n\geq 0} H_{\mathbb{C}}^{\otimes n}$ to be the Fock space with the deformed inner product

$$\langle h_1 \otimes \cdots \otimes h_m, k_1 \otimes \cdots \otimes k_n \rangle = \delta_{mn} \sum_{\sigma \in S_n} \varphi(\sigma) \prod_{j=1}^{n} \langle h_j, k_{\sigma(j)} \rangle$$

Let $\ell_\varphi(h)$ denote the left creation operator, (Toeplitz-type operator) and define

$$s_\varphi(h) = \ell_\varphi(h) + \ell_\varphi(h)^*$$

Let

$$\omega_\varphi(x) = \langle \Omega, x\Omega \rangle$$

where Ω is the vacuum vector.

Traciality

A straightforward calculation shows that

$$\omega(s(h_1)\dots s(h_m)) = \sum_{\nu\in P_2(m)} \psi(\nu) \prod_{\{i,j\}\in\nu} \langle h_i, h_j\rangle.$$

Let e_1, e_2, e_3, e_4 be orthonormal vectors and $s(e_j) = s_j$.

$\omega(s_4 s_3 s_2 s_1 s_3 s_4 s_2 s_1) = \langle e_1 \otimes e_2 \otimes e_3 \otimes e_4, e_3 \otimes e_4 \otimes e_2 \otimes e_1\rangle = \varphi(1423)$

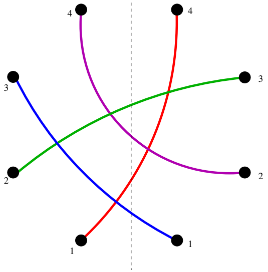

Figure: $\sigma = (1432)$

Rotated Once

$$\omega(s_1 s_4 s_3 s_2 s_1 s_3 s_4 s_2) = \langle e_2 \otimes e_3 \otimes e_4 \otimes e_1, e_1 \otimes e_3 \otimes e_4 \otimes e_2 \rangle = (14)$$

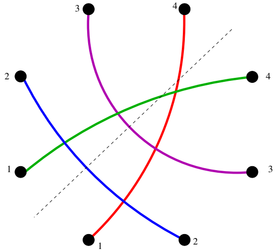

Figure: $\sigma = (14)$

Rotated Again

$$\omega(s_2 s_1 s_4 s_3 s_2 s_1 s_3 s_4) = \langle e_3 \otimes e_4 \otimes e_1 \otimes e_2, e_2 \otimes e_1 \otimes e_3 \otimes e_4 \rangle = \varphi(1324)$$

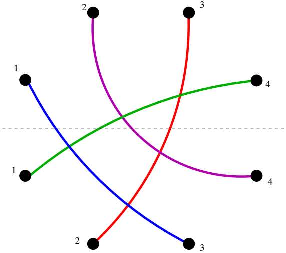

Figure: $\sigma = (1324)$

Traciality Conditions

Notice that

$$(1423) = (1432)(243) \text{ and } (14) = (1234)(132)$$

and that

$$(14) = (1432)(234) \text{ and } (1324) = (1234)(123)$$

So the conditions we need are

$$\varphi(\rho_n \iota_1(\sigma)) = \varphi(\rho_n^{-1} \iota_2(\sigma)) \quad (1)$$

$$\varphi(\iota_1(\sigma)) = \varphi(\iota_2(\sigma)) \quad (2)$$

where $\rho_{n+1} = (1, 2, \ldots, n+1)$, $\iota_1 : S_n \hookrightarrow S_{n+1}$ is the inclusion which stabilizes the last element, and ι_2 stabilizes the first.
Examples: $\varphi(\sigma) = q^{\iota(\sigma)}$ for $-1 \leq q \leq 1$ (Bożejko-Speicher 1991), $\varphi(\sigma) = q^{n-B(\sigma)}$ for $0 \leq q \leq 0$ (Bożejko-Speicher 1996).

A Theorem

Theorem (A.-Junge)

1. *Let* $H = L^2([0,\infty),\mathbb{R})$. $b_t := s_\varphi(\chi_{[0,t]})$ *is an exchangeable brownian motion if* φ *satisfies 1 and 2.*
2. *Let* (b_t) *be an exchangeable noncommutative brownian motion. There exists a positive definite function on* $\varphi_b : S_\infty \to \mathbb{R}$ *which satisfies 1 and 2 such that*

$$\tau(b_{I_n} \dots b_{I_1} b_{I_{\sigma(1)}} \dots b_{I_{\sigma(n)}}) = \varphi(\sigma)$$

Note: Since the real-valued positive definite functions which satisfy 1 and 2 are closed under pointwise multiplication and convex combinations, we can construct new Brownian motions from old ones using these operations.

Why isn't this everything?

Let

$$B_t = (b_{ij}(t))_{1 \leq i,j \leq N}$$

where $b_{ij}(t)$ are independent complex-valued brownian motions for $i \leq j$ and $b_{ij} = \bar{b_{ji}}$ for $i > j$.

In this case, $\psi(\nu) = N^{-g(\nu)}$ where g denotes the *genus* number of ν.

Let φ denote the restriction of ψ to permutations.

φ is positive definite and satisfies 1 and 2, so we may apply our theorem to obtain a brownian motion b'_t. However, b'_t is not the same brownian motion as B_t!

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Another Example (due to M. Guta)

Let $\psi_1, \ldots, \psi_m$ be functions on pair partitions which arise as moments of brownian motions $b_t^1, \ldots, b_t^m$.

For a pair partition $\nu \in P_2(2n)$, define

$$\psi_1 *_q \psi_2 *_q \cdots *_q \psi_n(\nu) = m^{-n} \sum_{c:\nu \to \{1,\ldots,m\}} q^{\iota(c,\nu)} \prod_j \psi_j(c^{-1}(j))$$

where $\iota(c,\nu) = \frac{1}{2}|\,(a,b)|a$ crosses $b, c(a) \neq c(b)\}\,|$.

For example, $\psi_{q_1} *_q \psi_{q_2}$ gives a brownian motion which does not come from the "naïve" construction.

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Thank You!