On the free gamma distributions

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Classical and free infinite divisibility

By $\mathcal{ID}(*)$ we denote the class of $*$-infinitely divisible probability measures on $\mathbb{R}$, i.e.

$$\mu \in \mathcal{ID}(*) \iff \forall n \in \mathbb{N}\ \exists \mu_n \in \mathcal{P}(\mathbb{R})\colon\ \mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ terms}}.$$

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Classical Lévy-Khintchine representation

Theorem [Lévy-Khintchine]. Let μ be a probability measure on $\mathbb{R}$ and consider its characteristic function

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Then μ is infinitely divisible, if and only if $\hat{\mu}$ has a representation in the form:

$$\log(\hat{\mu}(u)) = \mathrm{i}\eta u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\right)\, \rho(\mathrm{d}t).$$

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Here $\eta \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$, i.e.

$$\rho(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, t^2\}\, \rho(\mathrm{d}t) < \infty.$$

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The *characteristic triplet* (a, ρ, η) is uniquely determined.

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Theorem [Bercovici & Voiculescu]. Let μ be a probability measure on $\mathbb{R}$ with free cumulant transform

$$\mathcal{C}_\mu(z) = zG_\mu^{\langle -1 \rangle}(z) - 1, \qquad (z \in \mathcal{D}(\mu) \subseteq \mathbb{C}^-).$$

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The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mu \longleftrightarrow \log(\hat{\mu}(u)) = \mathrm{i}\eta u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\right)\, \rho(\mathrm{d}t)$$

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Direct formula: For any measure μ in $\mathcal{ID}(*)$ we have

$$\mathcal{C}_{\Lambda(\mu)}(\mathrm{i}z) = \int_0^\infty \log(\hat{\mu}(zx))\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (z < 0).$$

Properties of the Bercovici-Pata bijection

(i) If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.

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(iv) For measures $\mu, \mu_1, \mu_2, \mu_3, \dots$ in $\mathcal{ID}(*)$, we have

$$\mu_n \xrightarrow{\text{w}} \mu \iff \Lambda(\mu_n) \xrightarrow{\text{w}} \Lambda(\mu).$$

Examples.

(1) Let μ be the standard *Gaussian distribution*, i.e.

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Then $\Lambda(\mu)$ is given by

$$\begin{cases} (1-\lambda)\delta_0 + \frac{1}{2\pi t}\sqrt{(t-a)(b-t)}\cdot 1_{[a,b]}(t)\,\mathrm{d}t, & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{2\pi t}\sqrt{(t-a)(b-t)}\cdot 1_{[a,b]}(t)\,\mathrm{d}t, & \text{if } \lambda \geq 1, \end{cases}$$

where $a = (1-\sqrt{\lambda})^2$ and $b = (1+\sqrt{\lambda})^2$.

The free gamma distributions

It is not hard to show that

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- What is Λ(exponential distribution)?

The free exponential distribution

The classical exponential distribution $\mu(\mathrm{d}x) = \mathrm{e}^{-x} 1_{(0,\infty)}(x)\,\mathrm{d}x$ has cumulant function

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Setting $\nu = \Lambda(\mu)$ we then have for z in $(-\infty, 0)$ that

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\mathcal{C}_\nu(\mathrm{i}z) &= \int_0^\infty \log(\hat{\mu}(zx))\mathrm{e}^{-x}\,\mathrm{d}x \\
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Setting $z = \frac{1}{w}$ we find for w in $\mathbb{C}^+$ that

$$\begin{aligned}\mathcal{C}_\nu(1/w) &= \int_0^\infty \frac{\mathrm{e}^{-t}}{t}\Big(\frac{1}{1 - t/w} - 1\Big)\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{e}^{-t}}{t}\Big(\frac{w}{w-t} - 1\Big)\,\mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{e}^{-t}}{t}\Big(\frac{t}{w-t}\Big)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\Big(\frac{1}{w-t}\Big)\,\mathrm{d}t \\ &= G_\mu(w).\end{aligned}$$

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so that

$$G_\nu^{\langle -1\rangle}(\tfrac{1}{w}) = w + wG_\mu(w) = w + \int_0^\infty \frac{w}{w-t}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad (w \in \mathbb{C}^+).$$

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It follows from De la Vallé Poussin's Theorem that

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In addition we have that

$$\rho(\mathrm{d}t) = F_\mu'(t) 1_{D_1}(t)\, \mathrm{d}t.$$

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In particular the singular part σ of μ is concentrated on the set

$$\big\{x \in \mathbb{R} \;\big|\; \lim_{y\downarrow 0}|G_{\nu_\alpha}(x+\mathrm{i}y)| = \infty\big\}.$$

A fundamental lemma of Bercovici & Voiculescu

For any positive number δ, put

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If $\lim_{z\to 0, z\in\Gamma} u(z) = \ell$, then $\lim_{z\to 0, z\in\triangle_\delta} u(z) = \ell$ for any positive number δ.

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so that

$$\frac{1}{w} = G_{\nu}(w + wG_{\mu}(w))$$

for all w in $\mathbb{C}^+$, such that $w + wG_{\mu}(w) \in \mathbb{C}^+$.

The curve: $\int_0^\infty \frac{t\mathrm{e}^{-t}}{(t-x)^2+y^2}\,\mathrm{d}t = 1.$

Let c_0 be the positive constant determined by

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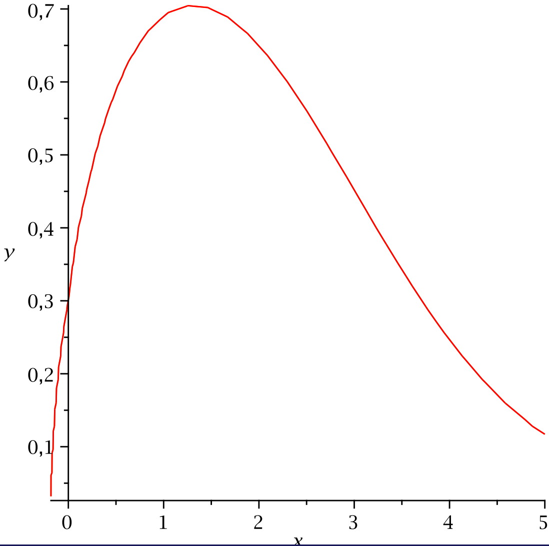
For any x in $[-c_0, \infty)$ there is a unique positive number $y = v(x)$, such that

$$\int_0^\infty \frac{t\mathrm{e}^{-t}}{(t-x)^2+y^2}\,\mathrm{d}t = 1.$$

The curve: $\int_0^\infty \frac{te^{-t}}{(t-x)^2+y^2}\,\mathrm{d}t = 1.$



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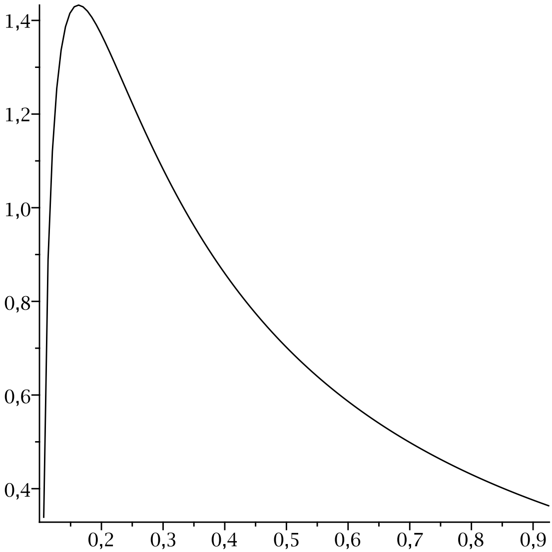

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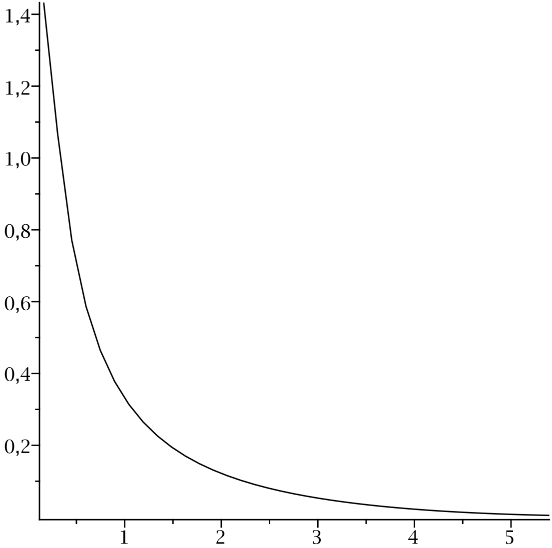

Asymptotic behavior of the free exponential distribution

The asymptotic behavior of $f_\nu(\xi)$ as $\xi \to \infty$ is given by

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At the lower bound $s_0 := \inf \operatorname{supp}(\nu) = P(-c_0)$, we have that

$$f_\nu(\xi) = \frac{\sqrt{2}}{\pi c_0 \sqrt{s_0 - c_0^2}}(\xi - s_0)^{1/2} + o(\xi - s_0), \quad \text{as } \xi \downarrow s_0.$$

Unimodality

A measure μ on $\mathbb{R}$ is called *unimodal*, if, for some a in $\mathbb{R}$, it has the form

$$\mu = \mu(\{a\})\delta_a + f(x)\,\mathrm{d}x,$$

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Theorem [Haagerup+T '11] The free gamma distributions are unimodal.

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Hence we want to show that

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$$C_\rho = \left\{ \tfrac{1}{\pi\rho} \sin(\theta) \mathrm{e}^{\mathrm{i}\theta} \;\middle|\; \theta \in (0, \pi] \right\}.$$

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By changes of variables and differentiation under the integral sign, one may verify that the function $\theta \mapsto F\left(\tfrac{1}{\pi\rho} \sin(\theta) \mathrm{e}^{\mathrm{i}\theta}\right)$ is strictly decreasing on $(0, \theta_0]$ and strictly increasing on $[\theta_0, \pi]$ for some θ_0.

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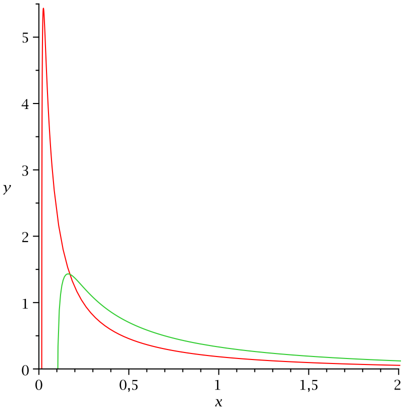
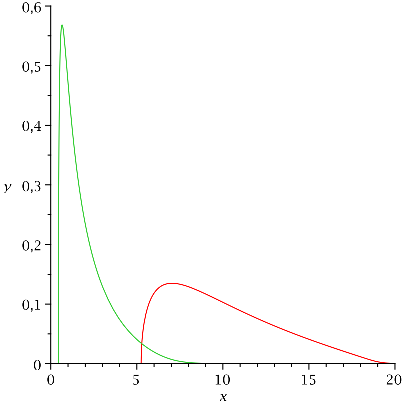
Graphs of f_α for $\alpha = \frac{1}{2}, 1, 2, 10$.





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(i) For any p in $\mathbb{N}$ we have that

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Proof of convergence in moments

The (classical) cumulant transform of μ_α is given by

$$\log(\hat{\mu}_\alpha(u)) = \alpha \int_0^\infty (\mathrm{e}^{\mathrm{i}ut} - 1)\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = \alpha \sum_{p=1}^\infty \frac{\mathrm{i}^p (p-1)!}{p!} u^p.$$

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$$\frac{1}{\alpha}\int_0^\infty t^p\,\nu(\mathrm{d}t) \underset{\alpha\to 0}{\longrightarrow} (p-1)! = \int_0^\infty t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$