Cauchy-Stieltjes kernel families

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NEF versus CSK families

The talk will switch between two examples of kernel families

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▶ Natural exponential families (NEF):

\[ P_\theta(dx) = \frac{1}{L(\theta)} e^{\theta x} \mu(dx) \]

\( \mu \) is a \( \sigma \)-finite measure, \( \Theta = (\theta_-, \theta_+) \).
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- Natural exponential families (NEF):
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  \( \mu \) is a \( \sigma \)-finite measure, \( \Theta = (\theta_-, \theta_+) \).

- Cauchy-Stieltjes kernel families (CSK):
  \[ P_\theta(dx) = \frac{1}{L(\theta)} \frac{1}{1 - \theta x} \mu(dx) \]
  \( \mu \) is a probability measure with support bounded from above.
  The "generic choice" for \( \Theta \) is \( \Theta = (0, \theta_+) \).
A specific example of CSK
Noncanonical parameterizations

Let $\mu = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1$ be the Bernoulli measure
> ”Noncanonical” parametrization:

$P_\theta = 1 - \theta^2 - \theta \delta_0 + \frac{1}{2} - \theta \delta_1$, $\theta \in (-\infty, 1)$

$Q_p = \frac{1}{2} - p^2 - p \delta_0 + p \delta_1$, $p \in (0, 1)$

Bernoulli family parameterized by probability of success $p$.

$p = \int_x Q_p (dx)$ (parametrization by the mean)
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- ”Canonical” parametrization: $p = \frac{1}{2-\theta}$

- $Q_p := P_{2-\frac{1}{p}} = (1 - p) \delta_0 + p \delta_1$, $p \in (0, 1)$
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- Bernoulli family parameterized by probability of success $p$.
  $$p = \int xQ_p(dx) \text{ (parametrization by the mean)}$$

Parametrization by the mean

\[ m(\theta) = \int xP_\theta(dx) = \begin{cases} 
\frac{L'(\theta)}{L(\theta)} & \text{NEF} \\
\frac{L(\theta)-1}{\theta L(\theta)} & \text{CSK}
\end{cases} \]

- For non-degenerate measure \( \mu \), function \( \theta \mapsto m(\theta) \) is strictly increasing and has inverse \( \theta = \psi(m) \).
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- \( \theta \mapsto m(\theta) \) maps \( (0, \theta_+) \) onto \( (m_0, m_+) \), "the domain of means".
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- Parameterizations by the mean:

\[ \mathcal{K}(\mu) = \{ Q_m(dx) : m \in (m_0, m_+) \} \]

where \( Q_m(dx) = P_{\psi(m)}(dx) \)
Variance function

\[ V(m) = \int (x - m)^2 Q_m(dx) \]

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- Variance function \( V(m) \) (together with the domain of means \( m \in (m_-, m_+) \)) determines NEF uniquely (Morris (1982)).
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- Variance function \( V(m) \) (together with \( m_0 = m(0) \in \mathbb{R} \), the mean of \( \mu \)) determines measure \( \mu \) uniquely (hence determines CSK uniquely).
Example: a CSK with quadratic variance function

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- The variance function is $V(m) = m(1 - m)$
- The generating measure $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ is determined uniquely once we specify its mean $m_0 = 1/2$.

That is, there is no other $\mu$ that would have mean 1/2 and generate CSK with variance function $V(m)$ that would equal to $m(1 - m)$ for all $m \in (1/2 - \delta, 1/2 + \delta)$
All NEF with quadratic variance functions are known
Morris class. Meixner laws

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- Letac-Mora (1990): cubic $V(m)$
- Various other classes Kokonendji, Letac, ...
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Suppose $m_0 = 0$, $V(0) = 1$.

Theorem (WB.-Ismail (2005))

1. $\mu$ is the Wigner's semicircle (free Gaussian) law iff $V(m) = 1$
2. $\mu$ is the Marchenko-Pastur (free Poisson) type laws
3. $\mu$ is the “free Gamma” type law iff $V(m) = (1 + bm)^2$
4. $\mu$ is the free binomial type law (Kesten law, McKay law) iff $V(m) = 1 + am + bm^2$ with $-1 \leq b < 0$
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Reproductive properties of NEF and CSK

Theorem (NEF: Jörgensen (1997))

If \( \mu \) is a probability measure in NEF with variance function \( V(m) \), then for \( r \in \mathbb{N} \) the \( r \)-fold convolution \( \mu_r := \mu^*r \), is in NEF with variance function \( rV(m/r) \).

Note

▶ If \( rV(m/r) \) is a variance function for all \( r \in (0, 1) \) then \( \mu \) is infinitely divisible.
▶ The domains of means behave differently.
▶ The ranges of admissible \( r \geq 1 \) are different.
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If a probability measure $\mu$ generates CSK with variance function $V_{\mu}(m)$, then the free additive convolution power $\mu_r := \mu \boxplus r$ generates the CKS family with variance function $rV_{\mu}(m/r)$.

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Pseudo-Variance function for CSK

- The variance

\[ V(m) = \frac{1}{L(\psi(m))} \int \frac{(x - m)^2}{1 - \psi(m)x} \mu(dx) \]

is undefined if \( m_0 = \int x \mu(dx) = -\infty \). (This issue does not arise for NEF)
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- When \( V(m) \) exists, consider

\[ \nabla(m) = \frac{m}{m - m_0} V(m) \]
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\nabla(m) = m \left( \frac{1}{\psi(m)} - m \right)
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(1)

where \( \psi(\cdot) \) is the inverse of \( \theta \mapsto m(\theta) = \int xP_\theta(dx) \) on \((0, \theta_+)\).
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where \( \psi(\cdot) \) is the inverse of \( \theta \mapsto m(\theta) = \int x P_\theta(dx) \) on \((0, \theta_+)\).

Expression (1) defines a ”pseudo-variance” function \( \nabla(m) \) that is well defined for all non-degenerate probability measures \( \mu \) with support bounded from above.
Properties of pseudo-variance function

- Uniqueness: measure $\mu(dx)$ is determined uniquely by $\nabla$

Explicit formula for the CSK family:

$$Q_m(dx) = L(\psi(m))(1 - \psi(m)x)\mu(dx) = V(m) + m(m - x)\mu(dx)$$

Reproductive property still holds

Theorem (WB-Hassairi (2011))

Let $V_\mu$ be a pseudo-variance function of the CSK family generated by a probability measure $\mu$ with support bounded from above and mean $-\infty \leq m_0 < \infty$. Then for $m > r m_0$ close enough to $r m_0$,$$V_\mu \boxplus r(m) = r V_\mu(m/r).$$ (2)
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**Theorem (WB-Hassairi (2011))**

Let \( \nabla_\mu \) be a pseudo-variance function of the CSK family generated by a probability measure \( \mu \) with support bounded from above and mean \(-\infty \leq m_0 < \infty\). Then for \( m > rm_0 \) close enough to \( rm_0 \),

\[
\nabla_\mu \boxtimes r(m) = r \nabla_\mu(m/r).
\]
Example: CKS family with cubic pseudo-variance function

Measure $\mu$ generating CSK with $\nabla (m) = m^3$ has density

$$f(x) = \frac{\sqrt{-1 - 4x}}{2\pi x^2} 1(-\infty, -1/4)(x)$$  \hspace{1cm} (3)

From reproductive property it follows that $\mu$ is $1/2$-stable with respect to $\boxplus$, a fact already noted before: [Bercovici and Pata, 1999, page 1054], [Pérez-Abreu and Sakuma, 2008]

$$Q_m(dx) = \frac{m^2 \sqrt{-1 - 4x}}{2\pi (m^2 + m - x)x^2} 1(-\infty, -1/4)(x) dx : m \in (-\infty, m_+)$$

What is $m_+$?

End now
Domain of means: $\{ Q_m : m \in (m_0, m_+) \}$

For $\nabla(m) = m^3$ the domain of means is $(-\infty, m_+)$, where:

1. $\theta \mapsto m(\theta)$ is increasing, so $m_+ = \lim_{\theta \uparrow \theta_{\text{max}}} m(\theta)$. This gives $m_+ = -1$
Domain of means: \( \{ Q_m : m \in (m_0, m_+) \} \)

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2. \( \frac{1}{1-\theta x} 1_{(-\infty,-1/4)}(x) \) is positive for \( \theta \in (0, \infty) \cup (-\infty, -4) \). The domain of means can be extended to \( m_+ = \lim_{\theta \to -4} m(\theta) \). This extends the domain of means up to \( m_+ = -1/2 \)

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3. \( \frac{m^2}{m^2 + m - x} 1_{(-\infty,-1/4)}(x) \) is positive for \( m \neq -1/2 \).
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   ▶ But \( \int Q_m(dx) < 1 \) for \( m > 1/2 \).
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3. \( \frac{m^2}{m^2 + m - x} 1_{(-\infty, -1/4)}(x) \) is positive for \( m \neq -1/2 \).

- But \( \int Q_m(dx) < 1 \) for \( m > 1/2 \).
- \( Q_m(dx) = \frac{m^2}{(m^2 + m - x)} \mu(dx) + \frac{(1+2m)^2}{(m+1)^2} \delta_{m+m^2} \) is well defined and parameterized by the mean for all \( m \in (-\infty, \infty) \).
Summary

Kernels $e^{\theta x}$ and $1/(1 - \theta x)$ generate NEF and CSK families

Similarities

- Parameterizations by the mean
- Quadratic variance functions determine interesting laws
- Convolution affects variance function for NEF in a similar way as the additive free convolution affects the variance function for CSK

Differences

- The generating measure of a NEF is not unique.
- A CSK family in parameterizations by the mean may be well-defined beyond the "domain of means"
- For CSK family, the variance function may be undefined. Instead of the variance function [Bryc and Hassairi, 2011] look at the "pseudo-variance" function $m \mapsto mV(m)/(m - m_0)$ which is well defined for more measures $\mu$. 
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Kernels $e^{\theta x}$ and $1/(1 - \theta x)$ generate NEF and CSK families

Similarities

- parameterizations by the mean
- Quadratic variance functions determine interesting laws
- Convolution affects variance function for NEF in a similar way as the additive free convolution affects the variance function for CSK

Differences

- The generating measure of a NEF is not unique.
- A CSK family in parameterizations by the mean may be well defined beyond the “domain of means”
- For CSK family, the variance function may be undefined. Instead of the variance function [Bryc and Hassairi, 2011] look at the "pseudo-variance" function $m \mapsto mV(m)/(m - m_0)$ which is well defined for more measures $\mu$. 
Thank you
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Stable laws and domains of attraction in free probability theory.
With an appendix by Philippe Biane.
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Free exponential families as kernel families.

One-sided Cauchy-Stieltjes kernel families.
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