Part I: Correlation Risk and Common Methods

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Outline

- Origins of Correlation Risk in Energy Trading
- Basic Concepts and Notation
- Temporal Correlation
  - Stack-and-Roll Hedging
  - Forward Yields, Inventory and Forward Dynamics
- Common Modeling Approaches
  - Comment on Spread Options
  - Reduced Form Models
  - Econometric and Structural Models
Correlation Risk—Hedging

- **Correlation risk most commonly arises in practice in hedging simple illiquid positions with simple liquid positions.**
  
  - Stack-and-roll hedging (Temporal):
    Long-tenor risk hedged with short-tenor instruments.
  
  - Basis (Locational):
    Hedging risks at illiquid delivery locations with liquid delivery locations.
  
  - Cross-commodity (Conversion):
    Hedging commodities with limited liquidity in swaps markets with closely related proxies having liquid markets.
Correlation Risk—Assets and Structured Transactions

- Most physical assets and structured commodities hedges of such involve the concept of transforming one commodity/delivery location/delivery time to another.
  - Storage (Temporal):
    Storing a commodity “now” for delivery “then.”
  - Transport (Locational):
    Moving a commodity from a supply source to a demand sink.
  - Refining and Generation (Conversion):
    Transforming one commodity to another commodity or set thereof.
Correlation Risk—Customer Demand

- Demand/Price Risk
  - Demand for a quantity can vary over both short and long time-scales creating inherent correlation risks to the commodities supply chain.
  - Serving customers involves a settlement payoff of:
    \[
    \sum \left[ \tilde{D}_n (p_f - p_n) + (D_n - \tilde{D}_n) (p_f - p_n) \right]
    \]
    where:
    - $D_n$ is the demand in time period $n$.
    - $\tilde{D}_n$ is the expected demand;
    - $p_n$ is the spot price for period $n$.
    - $p_f$ is the fixed (contract) price.
Origins

Correlation Risk—Customer Demand

- Demand/Price Risk
  - Correlation between demand and price is always against the holder of the short position.
  - The following plots shows hourly spot power prices versus demand in PJM.
Correlation Risk—Commercial Operations

- Liquidity and Credit Risk

  - Whether exchange traded or OTC, hedging activities are usually accompanied by collateral posting requirements.
  
  - "Macro" relationships between demand and price on long time scales can cause substantial mismatches in collateral posting terms.

  - Example: Retail energy companies
    
    - Provide commodities to retail end-users (who typically are not margined)
    
    - Hedge this inherent short position via standard futures or OTC swaps markets (which are margined)
    
    - This mismatch in credit support can result in lethal collateral calls in highly volatile times.
Correlation Risk—Commercial Operations

- **Liquidity and Credit Risk**
  - The following plots shows the rolling cal strip for NYMEX WTI, NG and PJM power prices.
  - The collateral calls against entities with long energy hedges put on in mid-2008 were onerous.
Basic Concepts and Notation

Forward Curves

- The primitive underlying for commodities valuation is the term structure of prices for future delivery.
  - This figure shows the forward curve for WTI on 15Jan2009.
  - Each point represents the price for WTI delivered in subsequent months as of this pricing date.
Basic Concepts and Notation

Forward Curves

- WTI forward curve at a variety of dates.
  - Note the range of prices as well as the changes in the monotonicity
Basic Concepts and Notation

Forward Curves

- NG forward curve at a variety of dates.
  - Seasonality is superimposed on macro structure.
  - Note the breakdown from the WTI price levels in recent years.
Temporal Correlation

Stack-and-Roll Hedging

- Consider a long position established on 31 May 2011 of 2/day (2 lots per calendar day) of Henry Hub natural gas for financial settlement over the term Jan 12 to Dec 18.

- The delta profile below is (arguably) the optimal Cal 12 hedge.
Temporal Correlation

Stack-and-Roll Hedging

- Hedging long-tenor risk with short-tenor positions is referred to as “stack-and-roll” hedging.
  - The short-tenor hedge is “stacked” against the long-tenor positions.
  - As circumstances permit the hedge is “rolled” forward by unwinding the short-tenor position when long-tenor hedges can be established.

- The situation in commodities is fundamentally different than in rates.
  - Rates: Liquidity spans tenors with highly liquid swaps or bonds in 2’s, 5’s, 10’s and 30’s.
  - Commodities: Liquidity is always concentrated at short tenors.

- It is hard to neutralize multiple-factors.
Temporal Correlation
Stack-and-Roll Hedging

- WTI open-interest by contract:
Temporal Correlation

Stack-and-Roll Hedging

- Why did we choose to sell -6.75/day of the Cal12 strip?

- Notation:
  - Denote the Cal12 strip by weights $\mathbf{w}$ corresponding to the exact volumes per month of the calendar strip (1/day).
  - Denote the original 2/day purchase over the seven-year strip by $\mathbf{W}$.
  - Let $\mathbf{w}^*$ and $\mathbf{W}^*$ denote these respective weights multiplied component-wise by the discount factors to the contract settlement dates.

- The minimum variance problem becomes:

$$
\min_{\alpha} \text{var} \left[ \alpha \left( \mathbf{w}^* d\bar{F} \right) + \mathbf{W}^* d\bar{F} \right]
$$

(1)

- The unknown $\alpha$ is the optimal quantity of the Cal12 hedge.
Temporal Correlation

Stack-and-Roll Hedging

- Defining the vectors obtained from component-wise multiplication:

\[ \vec{\eta} \equiv \vec{w} \ast \vec{F} \quad \text{and} \quad \vec{\zeta} \equiv \vec{W} \ast \vec{F} \]

the solution is:

\[ \alpha = -\frac{\vec{\eta}^\dagger A \vec{\zeta}}{\vec{\eta}^\dagger A \vec{\eta}} \] (2)

where:

- \( A \) is the matrix of the returns covariance between the set of contract months spanning the problem.

- The numerator is the covariance between the risk that we want to hedge \( \vec{W}^\dagger d\vec{F} \) and the risk of the strip that we will use to effect the hedge \( \vec{w}^\dagger d\vec{F} \).

- The denominator is the variance of the hedge value.
Temporal Correlation

Stack-and-Roll Hedging

- Returns PCA:

\[ r(t, \bar{T}_n) = \sum_{j} \lambda_j^{\frac{1}{2}} \Phi_j (\bar{T}_n - t) Z_j(t) \]

where \( \{\lambda_j, \Phi_j\} \) are the eigenvalues/eigenvectors of the covariance matrix of the returns series.

- The next figure shows the results from PCA analysis of returns of the first 36 nearby series 2002 through 2012.
  - The top plot shows \( \sqrt{\lambda_j} \) to put things into factor standard deviations.
  - The first two factors comprise over 99% of the variance.
  - The first factor decays with tenor as expected; the second factor also exhibits the standard structure of having one sign change, this occurring at a tenor of approximately 1 year.
Temporal Correlation
Stack-and-Roll Hedging

![Graphs showing WTI: Spectrum (Factor Vol) and WTI: Factors with 1st, 2nd, and 3rd factors]

0
0.02
0.04
0.06
0.08
0.1

WTI: Spectrum (Factor Vol)

0 5 10 15 20 25
−0.5
0
0.5
1

WTI: Factors

1st Factor
2nd Factor
3rd Factor
If we assume that the first PCA factor is the only driver, the minimum-variance problem reduces to:

$$\alpha = -\frac{\zeta^\dagger \Phi_1}{\eta^\dagger \Phi_1}$$

(3)

where $\Phi_1$ is the first PCA factor.

Equivalently:

$$0 = [\alpha \eta + \zeta]^\dagger \Phi_1$$

The result is a portfolio that is orthogonal to the first factor.

This was the approach that we used to calculate the hedge shown earlier.
Temporal Correlation

Stack-and-Roll Hedging

- The PCA results used forwards with tenor of three years (liquid).
- The stack-and-roll problem required extrapolation.
  - Here we used a two-factor exponential fit with the results shown below
  - The extrapolated values were then used for $\Phi_1$

Key questions:
- How much risk are we sustaining due to the unhedged higher factors?
- How stable are the covariance statistics?
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- This figure shows the $L^2$ error of a double-exponential fit of empirical returns variance using 10-day returns for three different non-overlapping 4 year intervals with starting dates: Jan2001, Jan2005 and Jan2009.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- The following shows rolling 4 year estimate of the 1 year volatility normalized by the 0 year vol, indexed by end date of estimation interval.

- As this and the previous figures show, the variation in estimated backwardation is significant.
Temporal Correlation

Forward Yields, Inventory and Forward Dynamics

- The dynamics of the WTI forward curve has become increasingly one-dimensional in nature.

- In the next figure:
  - The top plot displays the ratio of the rolling one-year realized volatility for the 12th nearby contract returns to that of the 2nd nearby, with the results plotted versus the mid-point of the calendar averaging window.
  
  - The lower plot shows the ratio of higher order total volatility to the that of the first factor: \( \left( \sum_{j=2}^{24} \lambda_j \right)^{\frac{1}{2}} \lambda_1^{\frac{3}{2}} \) from PCA analysis of the first 24 nearby contracts, applied over the same rolling one-year windows.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- These phenomena have significant consequences for hedging strategy construction and model development.

WTI: Ratio of Returns Volatility (2nd and 12th Nearby Contracts)

WTI: Ratio of Higher Factor to First Factor Eigenvalues
Forward Yields, Inventory and Forward Dynamics

- Forward curves can be viewed as yield curves.
- Forward yield:

\[ y(t, T, T + S) = \frac{1}{S} \log \left[ \frac{F(t, T + S)}{F(t, T)} \right] \]

- The forward yield annualized rate implied by borrowing to buy the commodity at time \( T \) and sell it at time \( T + S \).
- Negative forward yields imply that market participants are willing to pay a premium for earlier delivery
  - This is effectively lending at negative rates.
  - This happens when supply is constrained.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- Yields often exhibit extreme values.
  - The following is the WTI forward curve and forward yield for $S = \text{one month}$ in early Jan2009.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- Seasonality yields negative forward yields consistently for seasonal commodities.
Temporal Correlation

Forward Yields, Inventory and Forward Dynamics

- For a consumption commodity arbitrage arguments yield an inequality:
  \[ F(t, T) \leq F(t, t)e^{[r(t, T)+q(t, T)](T-t)} \]

- \( r \) and \( q \) are funding and storage rates.
- One can always buy at the spot price and store to delivery at \( T \).
- The convenience yield provides the comfort of an seeing an equality:
  \[ F(t, T) = F(t, t)e^{[r(t, T)+q(t, T)-\eta(t, T)](T-t)} . \]

- Key Points:
  - All that can be ascertained from market data is \( q - \eta \).
  - The cost of storage is not exogenous.
    - Storage owners will charge what the market will bear.
    - The cost of storage is in reality a function of forwards and vols as opposed to an input.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- Incentives: The huge credit-crisis contango resulted in a massive increase in the use of VLCCs store oil and refined products.
- The figure shows the result outside of the Port of Singapore during Jan2009. (Source: Google Maps)
Temporal Correlation

Forward Yields, Inventory and Forward Dynamics

- Facilities exist to dampen the effects of anticipated (seasonal) and unanticipated demand fluctuations.
- Natural gas is particularly interesting.
- U.S. Natural Gas Markets
  - Annual gas consumption is roughly 25 Tcf with roughly 4 Tcf of imports.
  - Gas consumption is highly seasonal due to winter heating requirements.
  - Approximately 4 Tcf of natural gas storage facilitates accommodation of winter demand.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- North American demand is highly seasonal.
  - High winter peak demand and the relatively mild summer peaks (air conditioning power demand met with CC generation)
  - Non-seasonal production profile.
  - Recent increase in domestic production—shale gas glut.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- Roughly 4.2bcf of storage capacity resolves the production versus consumption mismatch.
- Compare historical inventory levels versus "normal".
  - "Normal" is a Fourier fit with the number of modes used determined by an out-of-sample selection method with estimates of working capacity.

U.S. Working Storage

![Graph showing U.S. Working Storage with Actual and Normal lines.](image-url)
Temporal Correlation

Forward Yields, Inventory and Forward Dynamics

- Storage Residual: \( R(t) \equiv S(t) - \bar{S}(t) \)

\[
\bar{S}(t) = \alpha + \beta t + \sum_{k=1}^{K} [\gamma_k \sin(2\pi kt) + \delta_k \cos(2\pi kt)]
\] (4)

U.S. Working Storage Residual
Temporal Correlation

Forward Yields, Inventory and Forward Dynamics

- Forward yields are related to the storage residual $R(t)$.
- The figure shows calendar strip forward yields versus storage residual.
  - Calendar strips are used to “strip out” seasonal effects.

![Graph showing NG 1st/2nd Cal Strip Carry Versus Inventory Residuals](image-url)
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

  - Note the higher prices, backwardation and greater seasonality in 2001.
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- This is a commonly observed phenomenon:
  - High forward yields (contango) incentivizes owners of storage to inject—this occurs when there is a surplus.
  - Negative forward yields (backwardation) encourages withdrawals—during times of scarcity.
  - The figure shows the forward yield between the first two cal strips of the WTI forward curve yield versus OECD crude oil stocks.

![WTI 1st/2nd Cal Strip Carry Versus Inventory](diag.png)

**WTI 1st/2nd Cal Strip Carry Versus Inventory**

- Contango (Carry > 0)
- Backwardation (Carry < 0)
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

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WTI 1st/2nd Cal Strip Carry Versus Inventory

- Contango (Carry > 0)
- Backwardation (Carry < 0)
Temporal Correlation
Forward Yields, Inventory and Forward Dynamics

- Lessons from copper:
  - A non-seasonal consumption commodity with credible inventory time series (LME).
  - The upper plot shows that copper inventories have spanned a considerable range over the past decade.
  - The lower is weekly averages of forward yields versus inventory.
Temporal Correlation

Forward Yields, Inventory and Forward Dynamics

- PCA on copper daily forward returns yields the usual factors.
  - The figure shows the volatility of factors 2 and above over the volatility of the first factor by month versus average inventory.
  - Note the apparent increase in higher factor contributions at low inventory.
  - This is one origin of the (apparent) non-stationarity of returns covariances.
Comment on Spread Options—Examples

- **Calendar Spread Options**
  - Example: A CSO straddle payoff takes the form:
    \[ [F(\tau, T_2) - F(\tau, T_1)]^+ \]
    where \( \tau \) denotes option expiry.
  - In general the payoff is a function of the spread between two contracts at expiry \( \tau \):
    \[ F(\tau, T_2) - F(\tau, T_1) - K. \]
  - CSO are closely related to physical storage and hedges thereof.

- **Swaptions**
  - The option payoff references a strip of contract prices:
  - For example a call swaption has the payoff:
    \[ \max \left[ \sum_m d(\tau, T_m) (F(\tau, T_m) - K), 0 \right] \]
Common Modeling Approaches

Comment on Spread Options—Examples

- **Tolling Deals/Heatrate Options**
  - A spread-option between power and a fuel, typically natural gas with payoff:
    \[
    \max \left[ F_B(\tau, \tau) - H_\ast G(\tau, \tau) - V, 0 \right]
    \]
    for a sequence of days indexed by $\tau$, where:
    - $F_B$ and $G$ denote the prices of power (delivery bucket $B$) and natural gas respectively.
    - $H_\ast$ is the heatrate and $V$ is as strike (unit cost of running).

- **Crack-Spread Options:**
  - Options on the spread between refined product and a reference crude oil prices:
    \[
    \max \left[ F_{Product}(\tau, T) - F_{crude}(\tau, T), 0 \right]
    \]
    - Product is usually heating oil or gasoline.
    - Both forwards in common units (e.g. $$/Barrel).
Comment on Spread Options—Margrabe

- The spot prices of two assets $X$ and $Y$ under the money-market EMM are modeled as two standard GBMs:

$$dX_t = rX_t dt + \sigma_X X_t dB^{(X)}_t$$
$$dY_t = rY_t dt + \sigma_Y Y_t dB^{(Y)}_t$$

where the correlation between the two BMs is $\rho$

- Consider the following spread option with value:

$$d(T)\tilde{E} \left[ \max (X_T - Y_T, 0) \right]$$

- $d(T)$ is the discount factor (assume that interest rates are deterministic).
- Note that all of the options just mentioned are of this form for zero strikes.
Comment on Spread Options—Margrabe

- The standard valuation approach is via change of numeraire.\(^1\)
  - In the \(Y\)-measure \(\tilde{E}_Y\) in which \(Y_t\) is the numeraire, all assets discounted by \(Y\) must be martingales.
  - Denote the value of the option by \(V(t, X_t, Y_t)\).
  - \(V\) must be an \(\tilde{E}_Y\) martingale:
    \[
    \frac{V(0, X_0, Y_0)}{Y_0} = \tilde{E}_Y \left[ \frac{V(T, X_T, Y_T)}{Y_T} \right]
    \]
  - This implies:
    \[
    V(0, X_0, Y_0) = Y_0 \tilde{E}_Y \left[ \max \left( \frac{X_T}{Y_T} - 1, 0 \right) \right]
    \]

---

\(^1\)See Carmon and Durrleman, Pricing and Hedging Spread Options, 2003.
Common Modeling Approaches

Comment on Spread Options—Margrabe

- We know that the ratio:

\[ R_T = \frac{X_T}{Y_T} = \frac{X_0}{Y_0} \]

\[ e^{\sigma_X B_X(T) - \rho \sigma_X \sigma_Y B_Y(T) + \text{"DriftTerms"}} \]

- Also: \( \sigma_X B_X(T) - \sigma_Y B_Y(T) \) is a normal random variable with variance:

\[ \hat{\sigma}^2 \equiv \text{var} [\sigma_X B_X(T) - \sigma_Y B_Y(T)] = T \left[ \sigma_X^2 + \sigma_Y^2 - 2 \rho \sigma_X \sigma_Y \right] \]

- Finally, a fact about log-normals

  - If \( R \) is log-normal with variance \( \hat{\sigma}^2 \) then:

\[ E \left[ \max(R - K, 0) \right] = E(R)N(d_1) - KN(d_2) \]

  - where:

\[ d_{1,2} = \log \left[ \frac{E(R)}{K} \pm \frac{1}{2} \hat{\sigma}^2 \right] \]

\[ \hat{\sigma} \]
Common Modeling Approaches

Comment on Spread Options—Margrabe

Assembling the facts we have:

\[ V(0, X_0, Y_0) = Y_0 \left[ \tilde{E}_Y \left( \frac{X_T}{Y_T} \right) N(d_1) - N(d_2) \right] \]

\[ = X_0 N(d_1) - Y_0 N(d_2) \]

where:

\[ d_{1,2} = \frac{\log \left( \frac{X_0}{Y_0} \right) + \pm \frac{1}{2} \hat{\sigma}^2}{\hat{\sigma}} \]

Note: This is (arguably) intuitive:

- The value of the option is as if we used Black with \( X \) as the underlying and strike \( K \) replaced with \( Y_0 \).
- There is no discounting as funding is embedded in the \( Y \) asset.
- The implied vol \( \hat{\sigma} \) is obtained from the returns variance of \( X - Y \).
Common Modeling Approaches

Comment on Spread Options—Margrabe

- In our context forwards are not spot assets and the result must be discounted.
  
  - Let \( X_t = F(t, T_1) \) and \( Y_t = F(t, T_2) \).
  
  - The previous valuation formula must include discounting:
    
    \[
    V(0, F(0, T_1), F(0, T_2)) = d(T) \left[ F(0, T_1)N(d_1) - F(0, T_2)N(d_2) \right]
    \]

  - This can be verified by direct integration (more later).

  - Note that if all vols are set to zero then the value of the option is intrinsic:
    
    \[
    V(0, F(0, T_1), F(0, T_2)) = d(T) \max \left[ F(0, T_1) - F(0, T_2), 0 \right]
    \]

- Analogous formulas hold for other standard European option payoffs.
Comment on Spread Options—Margrabe

- Greeks have similarly analogous forms:

\[
\frac{\partial V}{\partial X} = N(d_1) \quad \frac{\partial V}{\partial Y} = -N(d_2)
\]

and

\[
\Gamma = \frac{N'(d_1)}{\hat{\sigma}} \begin{pmatrix} \frac{1}{X} & -\frac{1}{Y} \\ -\frac{1}{Y} & \frac{X}{Y^2} \end{pmatrix}
\]

- \(\Gamma\) is positive-definite because:

\[
\bar{\alpha}^t \Gamma \bar{\alpha} = \frac{1}{X} \left[ \alpha_1 - \frac{X}{Y} \alpha_2 \right]^2
\]

- The spread option payoff is an option:
  - When \(\Delta\)-hedged, all directions point up.
Common Modeling Approaches

Reduced Form Models

- The term “reduced-form” refers to modeling frameworks which posit price dynamics in the absence of “fundamental” considerations.

- In commodities there are two meta-classes of reduced form models:
  - Spot/Convenience yield models: Model the joint behavior of spot price and convenience yields (and perhaps other variables).
  - Factor models: The HJM framework applied to commodities.

- These approaches are effectively functionally identical.
Common Modeling Approaches

Reduced Form Models—Spot/Convenience Yield Models

- "Schwartz-type" models and descendants.\(^2\)
- Explicit modeling of spot price dynamics with additional processes added.
  - These additional processes are typically convenience yields or long term price levels.
  - The original two-factor incarnation (Gibson and Schwartz):

\[
\begin{align*}
    dS_t &= (r_t - \delta_t) S_t dt + \sigma S_t dB^{(1)}_t \\
    d\delta_t &= \kappa (\theta - \delta_t) dt + \gamma dB^{(2)}_t
\end{align*}
\]

- Here we let \( S_t \equiv F(t, t) \).
- \( \delta_t \) is the instantaneous convenience yield. When \( \delta_t > 0 \) the spot price has a negative drift (net of financial carry \( r_t \)).

- Calibration to forwards requires finding time-varying drifts.

Common Modeling Approaches

Reduced Form Models—Gaussian Exponential Models

- Gaussian exponential framework:

\[
dF(t, T) = F(t, T) \sum_{j=1}^{J} \sigma_j(T) e^{-\beta_j(T-t)} dB_t^{(j)}
\]

- We have for now the form \( \sigma_j(T) \).
- Often the BMs are assumed independent for simplicity.

- Intuition (2-factor):

  - If \( \sigma_2 \equiv 0 \), this is a one-factor model identical to that described in the first section:

    \[
    \sigma(T-t) = \alpha e^{-\beta(T-t)}
    \]

  - The second factor will typically have \( \beta_2 \gg \beta_1 \) and is intended to represent shorter time-scale forward returns.
This is “HJM” for commodities introduced by Clewlow-Strickland.

Some useful facts:

- The integral of the returns for factor $j$ on contract $T$ over $[0, t]$ is:

$$
\sigma_j(T) \int_0^t e^{-\beta_j(T-s)} dB_s^{(j)} = \sigma_j(T)e^{-\beta_j(T-t)} \int_0^t e^{-\beta_j(t-s)} dB_s^{(j)}
$$

$$
= \sigma_j(T)e^{-\beta_j(T-t)} Y_j(s)
$$

where we have defined: $Y_j(t) = \int_0^t e^{-\beta_j(t-s)} dB_s^{(j)}$.

***This means that the distribution of returns for all tenors are simultaneously described by the processes $Y_j(t)$.

***This means that the dynamics of the entire forward curve are prescribe by a J-dimensional stochastic process.
Reduced Form Models—Gaussian Exponential Models

- **Returns are normally distributed** since any integral of the form \( \int_0^t \phi(u) dB_u \) is normally distributed with mean zero.

The variance is obtained by the Ito isometry:

\[
E \left[ \left( \int_0^t \phi(s) dB_s \right)^2 \right] = \int_0^t \phi^2(s) ds.
\]

Therefore, the returns variance for \( F(t, T) \) is:

\[
V(t, T) \equiv \sum_{j=1}^{J} \sigma_j^2(T) \int_0^t e^{-2\beta_j(T-s)} ds
\]
Common Modeling Approaches

Reduced Form Models—Gaussian Exponential Models

Some useful facts: (cont)

- Recalling that \( Y_j(t) = \int_0^t e^{-\beta_j(t-s)} dB_s^{(j)} \), differentiating with respect to \( t \) yields:

\[
dY_j = -\beta_j \left[ \int_0^t e^{-\beta_j(t-s)} dB_s^{(j)} \right] dt + dB_t^{(j)}
\]

or

\[
dY_j = -\beta_j Y_j dt + dB_t^{(j)}
\]

The \( Y \)'s mean-revert toward a mean of zero with mean-reversion rates specified by the \( \beta \)'s.

Diffusions of this form are called Ornstein-Uhlenbeck processes.

Properties:

- \( E[Y_t|Y_0] = Y_0 e^{-\beta t} \)
- \( \text{var}[Y_t|Y_0] = \frac{1-e^{-2\beta t}}{2\beta} \)
- \( \text{var}[Y_\infty] = \frac{1}{2\beta} \)
Common Modeling Approaches

Reduced Form Models—Gaussian Exponential Models

- Some useful facts: (cont)

  - The resulting form for $F(t, T)$ is:

    $$F(t, T) = F(0, T)e^{\sum_{j=1}^{J} \left[ \sigma_j(T)e^{-\beta_j(T-t)}Y_j(t) \right]} - \frac{1}{2} V(t, T)$$

  - To see this note that $F(t, T)$ is a martingale and that for any normal random variable $Z$: $E \left[ e^Z \right] = e^{\frac{1}{2}\sigma_Z^2}$.

  - Note that the previous calculation is nothing more than the exponential equivalent of the previous GBM integration:

    $$F_t = F_0 \ e^{-\frac{1}{2}\sigma^2 t + \sigma B_t}$$

    with $\sigma^2 t$ replaced by the appropriate exponential integrals.
Reduced Form Models—A Caricature Model

A simple caricature of spot (daily) prices processes is for the returns of the daily spot prices to be i.i.d. normal:

$$F(t, t) = F_m(T_m)e^{\zeta Z_t - \frac{1}{2}\zeta^2}$$

where

- $F_m$ denotes the contract month containing day $t$
- $Z_t$ is standard normal.
- $\zeta$ is the spot volatility.
Common Modeling Approaches

Reduced Form Models—A Caricature Model

Spot volatility is implied by the monthly and daily vols

\[ \zeta^2 = \bar{\sigma}_D^2 T_d - \bar{\sigma}_M^2 T_m \]

where:
- \( T_d \approx T_m + \frac{1}{24} \).
- \( \bar{\sigma}_M \) and \( \bar{\sigma}_D \) are implied vols for monthly and daily options.
- The following figure shows the ATM spot vol by contract month for PJM.
Common Modeling Approaches

Reduced Form Models—A Caricature Model

- For comparison the following figure shows historical spot volatility for PJM and TETM3.
  - The reference price is the BOM contract fixing so that spot returns are defined as \( \log \left( \frac{p_d}{F_m(T_e)} \right) \).
Common Modeling Approaches

Reduced Form Models—Advantages

- A “good” reduced form model is tractable by construction-and captures some features of forward dynamics.

- Models such as the GEM models above can match observed covariance structures reasonably well.
  - Allowing for the BMs to be correlated the two factor model implies a correlation surface:

  \[
  \rho(T, S) = \frac{\sigma_1^2 e^{-\beta_1 (T + S)} + \sigma_1 \sigma_2 \rho \left[ e^{-(\beta_1 T + \beta_2 S)} + e^{-(\beta_2 T + \beta_1 S)} \right] + \sigma_2^2 e^{-\beta_2 (T + S)}}{\sigma(T) \sigma(S)}
  \]

- Since the returns of the \( T \) contract can be written as:

  \[
  \sigma_1 e^{-\beta_1 T} \left[ dB_t^{(1)} + \frac{\sigma_2}{\sigma_1} e^{(\beta_1 - \beta_2) T} dB_t^{(2)} \right]
  \]

  the free parameters in \( \rho(T, S) \) are:
  - The difference in the decay rates \( \beta_2 - \beta_1 \).
  - The volatility ratio \( \lambda = \frac{\sigma_2}{\sigma_1} \).
  - The correlation \( \rho \)
Common Modeling Approaches

Reduced Form Models—Advantages

- Example: WTI returns from Jan2007 to Dec2010:
  
  - Minimizing the Frobenius norm of the difference of the empirical and model covariance matrices yields:
    - The optimal decay rates are: $\bar{\beta} = [0.106, 1.528]$
    - The estimated correlation between the factors is $\rho = 0.119$
Common Modeling Approaches

Reduced Form Models—Advantages

- **Separation of time-scales:**
  - In natural gas and power common tradeables reference annual, monthly and spot prices at the daily or hourly level daily.
  - This is commonly handled by a large range of mean-reversion rates ($\beta$’s) in models.
  - Typically $\beta_1 \in [.1, .5]$ and $\beta_2 \gg 10$ for a two-factor model.
  - Spot returns statistics give us some guidance to $\beta_2$.
  - A useful definition for spot returns: $\log \left( \frac{p_d}{F_m(T_e)} \right)$ where $F_m(T_e)$ denotes the forward price for the contract month at expiration.
What are spot returns for a non-storable commodity?

- The plot below shows both the ACF as well as the implied $\beta$ by lag.
- The implication is that very high mean-reversion rates are required for some factors.
- This is easily handled in the reduced-form framework.
Common Modeling Approaches

Reduced Form Models—Disadvantages

- Production implementations are almost always predicated on constant correlation parameters.
  - Covariances may evolve with local time $t$ due to term structure of volatility but underlying correlation parameters between the BMS are usually assumed to be constant.

- Correlation parameters are not static.
  - Inventory affects covariance structure and forward yield affect inventory.
  - In the case of power many input fuels effect market clearing prices and ultimately should effect forward correlation structure.

- There is rarely enough liquidity in vol and correlation tradables to facilitate a “robust” calibration of reduced-form models.
  - Statistical estimation is used to set many parameters.
  - Models are designed with tractability as a dominant consideration diminishing empirical relevance and limiting their utility as valuation extrapolators.
Common Modeling Approaches
Econometric and Structural Models

- Major Theme: Severe limitations in liquidity and spectrum of market tradables will render rote application of reduced form models problematic in many circumstances.

- Econometric and structural models are intended to produce more realistic price processes which in theory should function better for pricing and hedging given limited market data.

- Structural Models:
  - In the case of some energy commodities, a great deal is known about supply and demand and market mechanics.
  - Generator stacks and load dynamics in power.
  - Weather-driven demand dynamics and inventory in natural gas.
  - Models that start with a caricature of the underlying market mechanics, are referred to as structural models.
  - Stack models: In which power prices are set via load (often weather driven) clearing through a generator stack.
  - Inventory models: In which price clearing occurs in the presence of an inventory process.
Common Modeling Approaches
Econometric and Structural Models

- **Econometric Models:**
  - Craft regressions tailored to individual markets, often using similar stylized facts as for structural models.
  - The following figure shows:
    - Historical peak spot heatrates (the ratio of power to natural gas prices) for PJM (more later) versus KPHL temperature.
    - A regression relating expected heatrates versus temperature.
Common Modeling Approaches

Econometric and Structural Models

- Econometric Models:
  - Based on regressions of historical behavior of relevant underlying variables.
  - The results yield simulation methods to generate the joint distribution of future realizations of these variables.
  - These realizations yield physical measure distributions of:
    * The payoff $\Pi$ of whatever the structure is that you are valuing.
    * Available hedges $\vec{H}$ which trade at market prices $\vec{p}_H$.
  - Standard portfolio analysis method can then be applied—for example, construction of minimum variance hedges:

\[
\min \text{var} \left[ \Pi + \vec{w}^\dagger \left( \vec{H} - \vec{p}_H \right) \right]
\]
On Deck

- Tolling Deals
- Variable Quantity Swaps
- CSOs and Natural Gas Storage