Irreducible holomorphic symplectic manifolds. moduli

(K. Hulek: Lecture II)

0. Moduli of K3 surfaces

We recall

Strong Torelli theorem: Let (S,h) and (S',h') be two polarized K3 surface. If there is a Hodge isometry $\phi: H^2(S',\mathbb{Z}) \to H^2(S,\mathbb{Z})$ with $\phi(h') = h$, then there exists a unique isomorphism $f: S \to S'$ with $\varphi = f^*$.

Recall also that we can always find markings

$$\varphi: H^2(S,\mathbb{Z}) \xrightarrow{\cong} L_{K3} = 3U \oplus 2E_8(-1)$$

$$\varphi(h) = h_{2d} = e + df \in U \subset 3U \oplus 2E_8(-1).$$

In this case the period point

$$\omega(S,\varphi) = [\varphi(H^{2,0}(S))] \in \Omega_{L_{h_{2d}}} = \mathcal{D}_{L_{h_{2d}}} \amalg \mathcal{D}'_{L_{h_{2d}}} \quad (\dim 19)$$

where

$$L_{h_{2d}} = h_{2d}^{\perp} \cong 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle.$$

The strong Torelli theorem leads us to consider the group

$$O(L_{K3}, h_{2d}) = \{ g \in O(L_{K3});\ g(h_{2d}) = h_{2d} \} \subset O(L_{h_{2d}}).$$

In fact

$$O(L_{K3}, h_{2d}) = \tilde{O}(L_{h_{2d}}) = \{ g \in O(L_{h_{2d}});\ g|_{L^\vee_{h_{2d}}/L_{h_{2d}}} = \mathrm{id} \}.$$

Let

$$\tilde{O}^+(L_{h_{2d}}) = \{ g \in \tilde{O}(L_{h_{2d}});\ g(\mathcal{D}_{L_{h_{2d}}}) = \mathcal{D}_{L_{h_{2d}}} \} \underset{\text{index } 2}{\triangleleft} \tilde{O}(L_{2d}).$$

This group acts properly discontinuously on $\mathcal{D}_{L_{h_{2d}}}$.

Together with the surjectivity of the period domain we obtain

Theorem The quotient space

$$\mathcal{F}_{2d} = \tilde{O}^+(L_{h_{2d}}) \backslash D_{L_{h_{2d}}}$$

is the moduli space of (semi-) polarized K3 surfaces of degree 2d.

Remark (i) $\dim \mathcal{F}_{2d} = 19$

(ii) $\mathcal{F}_{2d}$ has only finite quotient singularities

(iii) $\mathcal{F}_{2d}$ is quasi-projective.

I. Torelli theorem for IHSM

One has

- Unobstructedness of deformations, locally Torelli } as in K3 case
- surjectivity of the period domain (Huybrechts) }

X, X': IHSM.

Definition: We say that $\Phi: H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is a parallel transport operator if there exists a smooth, proper flat family $\pi: \mathcal{X} \to B$, points $b, b' \in B$, isomorphisms $\alpha: X \to \mathcal{X}_b$, and $\beta: X' \to \mathcal{X}_{b'}$ and a continuous path $\gamma: [0,1] \to B$ with $\gamma(0) = b$, $\gamma(1) = b'$ such that

$$\Phi = (\alpha')^* \circ \Gamma \circ (\alpha^{-1})^*$$

where $\Gamma: H^2(\mathcal{X}_b, \mathbb{Z}) \to H^2(\mathcal{X}_{b'}, \mathbb{Z})$ is the parallel transport along the path γ.

Torelli Theorem for IHSM (Verbitsky, Markman): Let X, X' be IHSM

(1) If there exists an isomorphism $\Phi: H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ of integral Hodge structures, which is a parallel transport operator, then X and X' are bimeromorphic

(ii) If φ maps a Kähler class to a Kähler class then X and X' are isomorphic, in fact $\varphi = f^*$ for some $f: X' \cong X$.

Remark: Parallel transport operators preserve the Beauville form. Thus they define a subgroup $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$.

II. Moduli

We first fix discrete data:

$$\left.\begin{array}{l} \cdot\ 2n = \text{dimension of } X \\ \cdot\ L = \text{Beauville lattice} \\ \cdot\ c = \text{Fujiki invariant} \end{array}\right\} \ \nu = (2n, L, c).$$

Next we fix an $O(L)$-orbit of

- $h \in L$, h primitive, $h^2 > 0$.

Let

$$\mathcal{M}_{\nu,h} = \text{moduli space of polarized IHSM with these data}$$

Such a moduli space exists by Viehweg's theory. As before let

$$L_h = h^\perp \subset L, \quad \Omega_{L_h} = \mathcal{D}_{L_h} \amalg \mathcal{D}'_{L_h}$$

Let

$$O^+(L,h) = \{g \in O(L);\ g(h) = h,\ g \text{ fixes } \mathcal{D}_{L_h}\} \subset \tilde{O}^+(L_h).$$

Theorem: For every fixed component $\mathcal{M}'_{\nu,h}$ of $\mathcal{M}_{\nu,h}$ there exists a finite to one dominant morphism

$$\psi: \mathcal{M}'_{\nu,h} \to O^+(L,h) \backslash \mathcal{D}_{L_h}.$$

Using parallel transport operators fixing the polarization we can define a normal subgroup

$$\Gamma_{\mathrm{Mon}} \subset O^+(L,h)$$

Theorem: The map ψ factors through an open immersion

$$\begin{tikzcd} \mathcal{M}'_{K,h} \arrow[r, hook, "\tilde{\psi}"] \arrow[dr, "\psi"'] & \Gamma_{\mathrm{Mon}} \backslash \mathcal{D}_{L_h} \arrow[d] \\ & O^+(L,h) \backslash \mathcal{D}_{L_h} \end{tikzcd}.$$

Remarks. (i) Unlike in the K3 case we do not know the precise image, unless $X \sim S^{[2]}$.
(ii) $\mathcal{M}_{K,h}$ can have several components (Apostol).
(iii) In the K3 case $\Gamma_{\mathrm{Mon}}/\pm id = O^+(L,h)/\pm id$.

III. The Hilb^n case

$$X \sim_{\mathrm{def}} \mathrm{Hilb}^n S = S^{[n]}, \quad n \geq 2$$

Then the Beauville lattice is

$$L = 3U \oplus 2E_8(-1) \oplus \langle -2(n-1) \rangle =: L_{K3,2n-2}.$$

If $r \in L$, $r^2 = \pm 2$ then this defines a reflection

$$\sigma_r(x) = x - 2\frac{(x,r)}{(r,r)} r$$

Let

$$\mathrm{Ref}(L) = \langle \sigma_r, -\sigma_{r'} ; r^2 = -2, (r')^2 = 2 \rangle \subset O^+(L)$$

Let

$$\hat{O}(L) = \{ g \in O(L) ; g|_{L^\vee/L} = \pm id \}$$

Theorem (Markman) In the case of $X \sim_{\mathrm{def}} \mathrm{Hilb}^n S$ one has that

$$\mathrm{Mon}^2(X) \cong \mathrm{Ref}(L_{K3,2n-2}) = \hat{O}^+(L_{K3,2n-2}).$$

Remark: For the other examples of IHSM the group $\mathrm{Mon}^2(X)$ is conjecturally known.

We now specialize even further:

$$X \sim \mathrm{Hilb}^2 S$$

and we consider "split" polarizations $h \in L_{K3,2}$, $h^2 = 2d > 0$ with

$$L_h = h^\perp \cong 2U \oplus 2E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2d \rangle.$$

Theorem: The moduli spaces of split-polarised IHSM of type $S^{[2]}$ are irreducible and of general type for $d \geq 12$.

Remark: GHS, Apostol for irreducibility.

This can be done by modular forms.

L: lattice of signature $(2,n)$, $\Omega_L = \mathcal{D}_L \amalg \mathcal{D}_L'$.

$\Gamma < O(L)$ finite index.

Definition: A modular form w.r.t. Γ of weight k and (finite) character χ is a holomorphic function

$$F : \mathcal{D}_L^\bullet \to \mathbb{C} \qquad (\mathcal{D}_L^\bullet = \text{cone over } \mathcal{D}_L \subset \mathbb{P}(L_{\mathbb{C}}))$$

such that

(i) $F(tZ) = t^{-k} F(Z)$, $t \in \mathbb{C}^*$, $Z \in \mathcal{D}_L^\bullet$

(ii) $F(\gamma Z) = \chi(\gamma) F(Z)$

Remark: If weight $F = nk$, character $= (\det)^k$ then

$$\omega_F = F.(dZ)^k$$

is a Γ-invariant pluricanonical form.

Theorem (GHS). Assume $n \geq 9$. If there exists a cusp form F of weight $< n$ which vanishes along the reflection divisor with character $\det^\varepsilon$, $\varepsilon \in \{0,1\}$, then $\mathcal{F}_\Gamma$ is of general type.

Idea: Construct many pluricanonical forms which extend to a smooth projective model of $\mathcal{F}_\Gamma$. One has to take care of

1. Singularities
2. Heegner divisors
3. Obstructions from the boundary.

Construction of such forms

$$L_{2,26} = 2U \oplus 3E_8(-1) \cong 2U \oplus \Lambda \qquad (\Lambda = \text{Leech lattice})$$

On $\mathcal{D}_{L_{2,26}}$ one has the Borcherds modular form

$$\Phi_{12} : \mathcal{D}^{\bullet}_{L_{2,26}} \longrightarrow \mathbb{C} \qquad (\text{weight } 12).$$

Construct suitable embeddings

$$L_{h} \hookrightarrow L_{2,26} \rightsquigarrow \mathcal{D}_L \subset \mathcal{D}_{L_{2,26}}.$$

This can be done by first choosing a root $r_0 \in E_8(-1)$ and then consider embeddings

$$\langle -2d \rangle \mapsto \langle l_0 \rangle \subset (r_0)^{\perp}_{E_8(-1)} = E_7(-1) \subset E_8(-1). \qquad (l_0 = -2d)$$

Let

$$R_{l_0} := \{ r \in E_8(-1) \, ; \, r \perp l_0 \}.$$

$$N_{l_0} := |R_{l_0}|.$$

Then we define the quasi-pullback

$$\boxed{F_{l_0} := \frac{\Phi_{12}(Z)}{\prod_{\{\pm r\} \in R_{l_0}} (Z, r)} \Big|_{\mathcal{D}^{\bullet}_{L_{2d}}}}$$

We have

$$\text{weight } F_0 = 12 + \frac{k_e}{2} \overset{!}{<} 20$$

Hence we need an embedding with

$$1 \leq k_e \leq 7.$$

This leads to a number theoretic problem.

<u>Reference</u> Survey paper arXiv: 1005.4881