Boundaries of reduced C*-algebras of discrete groups

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$$\lambda : \ell^\infty(G) \to \mathbb{C},$$

i.e. a unital positive $G$-invariant linear map.
Definition

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In this case, $\lambda$ is a unital positive $G$-equivariant projection.
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Therefore, $G$ is non-amenable if $\mathbb{C}$ is “too small” to be the range of a unital positive $G$-equivariant projection on $\ell^\infty(G)$. 
Idea

Consider the minimal C*-subalgebra $A_G$ of $\ell^\infty(G)$ such that there is a unital positive $G$-equivariant projection

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The size of $A_G$ should somehow “measure” the non-amenability of $G$. 
Theorem (Kalantar-K 2014)

There is a unique minimal C*-algebra $\mathcal{A}_G$ arising as the range of a unital positive $G$-equivariant projection

$$P : \ell^\infty(G) \rightarrow \mathcal{A}_G.$$ 

The algebra $\mathcal{A}_G$ is isomorphic to the algebra $C(\partial_F G)$ of continuous functions on the Furstenberg boundary $\partial_F G$ of $G$. 
Motivation
Kirchberg proved that every exact C*-algebra can be embedded into a nuclear C*-algebra.
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In the separable case, Kirchberg and Phillips proved the nuclear C*-algebra can be taken to be the Cuntz algebra on two generators.
Ozawa conjectured the existence of what he calls a “tight” nuclear embedding.

**Conjecture (Ozawa 2007)**

Let $\mathcal{A}$ be an exact C*-algebra. There is a canonical nuclear C*-algebra $N(\mathcal{A})$ such that

$$\mathcal{A} \subset N(\mathcal{A}) \subset I(\mathcal{A}),$$

where $I(\mathcal{A})$ denotes the injective envelope of $\mathcal{A}$. 
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The algebra $N(\mathcal{A})$ will inherit many properties from $\mathcal{A}$, for example simplicity and primality.
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**Theorem (Ozawa 2007)**

Let $\mathcal{C}_r^*(\mathbb{F}_n)$ denote the reduced C*-algebra of $\mathbb{F}_n$ for $n \geq 2$. There is a canonical nuclear C*-algebra $N(\mathcal{C}_r^*(\mathbb{F}_n))$ such that

$$\mathcal{C}_r^*(\mathbb{F}_n) \subset N(\mathcal{C}_r^*(\mathbb{F}_n)) \subset I(\mathcal{C}_r^*(\mathbb{F}_n)),$$

where $I(\mathcal{C}_r^*(\mathbb{F}_n))$ denotes the injective envelope of $\mathcal{C}_r^*(\mathbb{F}_n)$. 
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**Theorem (Ozawa 2007)**

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where $I(C^*_r(\mathbb{F}_n))$ denotes the injective envelope of $C^*_r(\mathbb{F}_n)$.

Note that $C^*_r(\mathbb{F}_n)$ is exact since $\mathbb{F}_n$ is an exact group.
Ozawa takes $N(C^*_r(\mathbb{F}_n)) = C(\partial \mathbb{F}_n) \rtimes_r \mathbb{F}_n$, where $\partial \mathbb{F}_n$ denotes the hyperbolic boundary of $\mathbb{F}_n$. 
Ozawa takes $\mathcal{N}(C^*_r(F_n)) = C(\partial F_n) \rtimes_r F_n$, where $\partial F_n$ denotes the hyperbolic boundary of $F_n$.

**Key Proposition (Ozawa 2007)**

Let $\mu$ be a quasi-invariant doubly ergodic measure on $\partial G$. If

$$\varphi : C(\partial F_n) \to L^\infty(\partial G, \mu)$$

is a unital positive $F_n$-equivariant map, then $\varphi = \text{id}$. 
Equivariant Injective Envelopes
An *operator system* is a unital self-adjoint subspace of a C*-algebra.
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A G-operator system is an operator system equipped with the action of a group G, i.e. a unital homomorphism from G into the group of order isomorphisms on S.
Let $C$ be a category consisting of objects and morphisms. An object $I$ is injective in $C$ if, for every pair of objects $E \subset F$ and every morphism $\varphi : E \to I$, there is an extension $\tilde{\varphi} : F \to I$. When the objects are operator systems and the morphisms are unital completely positive maps, we get injectivity. When the objects are $G$-operator systems and the morphisms are $G$-equivariant unital completely positive maps, we get $G$-injectivity.
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The *$G$-injective envelope* of a $G$-operator system $S$ is the minimal $G$-injective operator system containing $S$. 
Theorem (Hamana 1985)

If $S$ is a $G$-operator system, then $S$ has a unique $G$-injective envelope $I_G(S)$. Every unital completely isometric $G$-equivariant embedding

$$\varphi : S \rightarrow T,$$

extends to a unital completely isometric $G$-equivariant embedding

$$\tilde{\varphi} : I_G(S) \rightarrow T.$$
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Since there is a unital completely isometric $G$-equivariant embedding of $S$ into $\ell^\infty(G, S)$ there are unital completely isometric $G$-equivariant embeddings

$$S \subset I_G(S) \subset \ell^\infty(G, S).$$
If $S$ is an operator system equipped with a $G$-action, then there are unital completely isometric $G$-equivariant embeddings

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and a unital positive $G$-equivariant projection $P : \ell^\infty(G, S) \to I_G(S)$. 
Upshot

If $S$ is an operator system equipped with a $G$-action, then there are unital completely isometric $G$-equivariant embeddings

$$S \subset I_G(S) \subset \ell^\infty(G, S),$$

and a unital positive $G$-equivariant projection $P : \ell^\infty(G, S) \to I_G(S)$.

The $G$-injective envelope $I_G(S)$ has a natural $C^*$-algebra structure (induced by the Choi-Effros product).
Corollary

Let $G$ be a discrete group acting trivially on $\mathbb{C}$ and let $I_G(\mathbb{C})$ denote the $G$-injective envelope of $\mathbb{C}$. Then

$$\mathbb{C} \subset I_G(\mathbb{C}) \subset \ell^\infty(G),$$

and there is a unital positive $G$-equivariant projection

$$P : \ell^\infty(G) \rightarrow I_G(\mathbb{C}).$$
**Corollary**

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The $G$-injective envelope $I_G(\mathbb{C})$ is a commutative $C^*$-algebra equipped with a $G$-action, so there is a compact $G$-space $\partial_H G$ such that $I_G(\mathbb{C}) \simeq C(\partial_H G)$.

We call $\partial_H G$ the **Hamana boundary of $G$**.
The Furstenberg Boundary
Definition

Let $X$ be a compact $G$-space.

1. The $G$-action on $X$ is *minimal* if the $G$-orbit

$$Gx = \{sx \mid s \in G\}$$

is dense in $X$ for every $x \in X$. 
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2. The $G$-action on $X$ is strongly proximal if, for every probability measure $\nu$ on $X$, the weak*-closure of the $G$-orbit

$$G\nu = \{s\nu \mid s \in G\}$$

contains a point mass $\delta_x$ for some $x \in X$. 
Definition (Furstenberg 1972)

A compact $G$-space $X$ is a *boundary* if it is minimal and strongly proximal.
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**Key Property**

If $X$ is a boundary, then for every probability measure $\nu$ on $X$, the weak*-$\sigma$-closure of the $G$-orbit $G\nu$ contains all of $X$.

Here $x \in X$ is identified with the point mass $\delta_x$ on $X$. 
The Hamana boundary $\partial_H G$ is a boundary in the sense of Furstenberg.
Theorem (Furstenberg 1972)

Every group $G$ has a unique boundary $\partial_F G$ that is universal, in the sense that every boundary of $G$ is a continuous $G$-equivariant image of $\partial_F G$. We refer to $\partial_F G$ as the Furstenberg boundary of $G$.
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We refer to $\partial_F G$ as the *Furstenberg boundary of $G$*.  

**Theorem (Kalantar-K 2014)**

*For a discrete group $G$, the Hamana boundary $\partial_H G$ can be identified with the Furstenberg boundary $\partial_F G$.***
Properties of injective envelopes (injectivity, rigidity and essentiality) imply corresponding results about the Furstenberg boundary.
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**Theorem (Kalantar-K 2014)**

Let $G$ be a discrete group and let $\partial_F G$ denote the Furstenberg boundary of $G$. Then the $C^*$-algebra $C(\partial_F G)$ is $G$-injective. Moreover, we have the following rigidity results:
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Let $G$ be a discrete group and let $\partial_F G$ denote the Furstenberg boundary of $G$. Then the C*-algebra $C(\partial_F G)$ is $G$-injective. Moreover, we have the following rigidity results:

1. Every unital positive $G$-equivariant map from $C(\partial_F G)$ is completely isometric.
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1. Every unital positive $G$-equivariant map from $C(\partial_F G)$ is completely isometric.

2. The only positive $G$-equivariant map from $C(\partial_F G)$ to itself is the identity map.
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3. If $M$ is a minimal $G$-space, then there is at most one unital $G$-equivariant map from $C(\partial_F G)$ to $C(M)$, and if such a map exists, then it is a unital injective *-homomorphism.
Exactness and Nuclear Embeddings
Definition (Kirchberg-Wasserman 1999)
A discrete group $G$ is exact if the reduced C*-algebra $C^*_r(G)$ is exact.
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**Theorem (Kalantar-K 2014)**

Let $G$ be a discrete group. Then $G$ is exact if and only if the $G$-action on the Furstenberg boundary $\partial_{\mathcal{F}} G$ is amenable.
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Applying a result of Anantharaman-Delaroche gives the following corollary.
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Applying a result of Anantharaman-Delaroche gives the following corollary.

**Corollary**

If $G$ is a discrete exact group, then the reduced crossed product $C(\partial_F G) \rtimes_r G$ is nuclear.
Theorem (Kalantar-K 2014)

Let $G$ be a discrete exact group. Then there is a canonical nuclear $C^*$-algebra $N(C^*_r(G))$ such that

$$C^*_r(G) \subset N(C^*_r(G)) \subset I(C^*_r(G)),$$

where $I(C^*_r(G))$ denotes the injective envelope of $C^*_r(G)$.
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Note: This is non-separable in general, but can be replaced by a separable nuclear $C^*$-algebra at the expense of no longer being canonical.
C*-Simplicity
Let $G$ be a discrete group. When is $G$ C*-simple, i.e. when is the reduced group C*-algebra $C^*_r(G)$ simple?
Open Problem

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Day showed in 1957 that every discrete group $G$ has a largest amenable normal subgroup $R_a(G)$ called the amenable radical of $G$. If $G$ is C*-simple, then $R_a(G)$ is necessarily trivial.
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Conjecture (de la Harpe, ?)

The reduced group C*-algebra $C^*_r(G)$ is simple if and only if the amenable radical $R_a(G)$ is trivial.
Let $G$ be a discrete group with identity element $e$. The $G$-action on a compact $G$-space $X$ is \textit{topologically free} if, for every $s \in G$, the set
\[
X \setminus X^s = \{ x \in X \mid sx \neq x \}
\]
is dense in $X$. 
The property of the $G$-action on the Furstenberg boundary $\partial_F G$ being topologically free is an intermediate property between C*-simplicity and triviality of the amenable radical $R_a(G)$. 

**Theorem (Kalantar-K 2014)**

1. If the $G$-action on $\partial_F G$ is topologically free, then $R_a(G)$ is trivial.
2. If $G$ is exact, and the reduced C*-algebra $C^*_r(G)$ is simple, then the $G$-action on $\partial_F G$ is topologically simple.
The property of the $G$-action on the Furstenberg boundary $\partial_F G$ being topologically free is an intermediate property between C*-simplicity and triviality of the amenable radical $R_a(G)$.

**Theorem (Kalantar-K 2014)**

Let $G$ be a discrete group.

1. If the $G$-action on $\partial_F G$ is topologically free, then $R_a(G)$ is trivial.
2. If $G$ is exact, and the reduced C*-algebra $\mathbb{C}^*_r(G)$ is simple, then the $G$-action on $\partial_F G$ is topologically simple.
**Figure:** Implications for an arbitrary discrete group $G$. 

- $C^*_r(G)$ simple
- $R_a(G)$ trivial
- $C(\partial_F G) \rtimes_r G$ simple
- $G \sim \partial_F G$ topologically free
Figure: Implications for a discrete exact group $G$. 

The diagram illustrates the relationships between various algebraic structures associated with $G$, including $C^*_r(G)$ being simple, $R_a(G)$ being trivial, $C(\partial_F G) \rtimes_r G$ being simple, and $G \sim \partial_F G$ being topologically free.
A Tarski monster group is a finitely generated group with the property that every nontrivial subgroup is cyclic of order $p$, for some fixed prime $p$. 

Theorem (Olshanskii 1982) Tarski monster groups exist for every prime $p > 10^{75}$: This answered a question of von Neumann about the existence of non-amenable groups which do not contain non-abelian free groups.
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**Theorem (Kalantar-K 2014)**

*If G is a Tarski monster group, then the G-action on the Furstenberg boundary $\partial_F G$ is topologically free.*
Rigidity of Maps
Theorem (Kalantar-K 2014)

Let $G$ be a non-amenable hyperbolic group, and let $\mu$ be an irreducible probability measure on $G$ with finite first moment. Let $\nu$ be a $\mu$-stationary probability measure on the hyperbolic boundary $\partial G$. If

$$\varphi : C(\partial G) \rightarrow L^{\infty}(\partial G, \nu)$$

is a unital positive $G$-equivariant map, then $\varphi = \text{id}$. 

We apply Jaworski's theory of strongly approximately transitive measures, combined with a uniqueness result of Kaimanovich for stationary measures.
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Corollary

Let $G$ be as above, and let $\partial_F G$ denote the Furstenberg boundary of $G$. Then

$$I_G(C(\partial G)) = C(\partial_F G),$$

where $I_G(C(\partial G))$ denotes the $G$-injective envelope of $C(\partial G)$. 

The Furstenberg boundary $\partial_F G$ can be thought of as a "projective cover" of the hyperbolic boundary $\partial G$. 
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Quantum Groups
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Many of our results hold in this setting. We intend to pursue this further...
Thanks!