1 Minimum number of unbordered factors — Kalle Saari

If a finite word $w$ of length $\geq 2$ is unbordered, there exist nonempty unbordered words $u$ and $v$ such that $w = uv$ [Ehrenfeucht and Silberger, Periodicity and unbordered segments of words, Discrete Math. 26 (1979), 101–109]. More precisely, letting $B(w)$ denote the number of distinct unbordered factors of $w$, define

$$b(n) = \min\{k \in \mathbb{N} : \text{there exists an unbordered word } w \text{ with } |w| = n \text{ and } B(w) = k\}$$

Experimental data [http://users.utu.fi/kasaar/unbordered.txt] suggest the following

**Conjecture.** If $w$ is unbordered and $B(w) = b(|w|)$, then $w$ is a Sturmian word.

An analogous question with unbordered words replaced by Lyndon words can be asked, and partial results in that direction have been obtained recently [K. Saari, Lyndon words and Fibonacci numbers, submitted. Preprint available at http://arxiv.org/abs/1207.4233].

2 Critical exponent of quasiperiodic words — Gwénaël Richomme

A *quasiperiodic* word is, here, a finite or infinite word that can be obtained by concatenations or overlaps of a given finite word, called a quasiperiod of the whole word. For instance, $aba$ is a quasiperiod of the Fibonacci word. It is an immediate observation that any quasiperiodic word contains an overlap. Thus its critical exponent is greater than 2. What is the minimal critical exponent of an infinite quasiperiodic word over an alphabet $A$?
3 Sums of digits of squares — Thomas Stoll

Let $s_2(n)$ denote the sum of the binary digits of the positive integer $n$.

**Question.** Let $k \in \{9, 10, 11, 14, 15\}$. Are there are infinitely many odd $n$ such that $s_2(n^2) = s_2(n) = k$ ?

**Remarks:**

- It is known that for $1 \leq k \leq 8$ there are only finitely many $n$, and that for $k \geq 16$, $k \in \{12, 13\}$ there are infinitely many $n$.

- This is the open question posed in the paper: K. Hare, S. Laishram, T. Stoll, The sum of digits of $n$ and $n^2$, *Int. J. Number Theory* 7 (2011), 1737–1759.

4 Palindromic factors — Luca Zamboni

**Prove or disprove:** Given an aperiodic infinite word $x$ and a positive integer $n$, there exists a finite factor $u$ of $x$ which is not a product of $n$ palindromes.

5 Total run lengths — Amy Glen

A *run* in a word is a periodic factor whose length is at least twice its period and which cannot be extended to the left or right without increasing the period. For example, the word $ababaabaa$ contains the following runs: $aa$ (twice), $ababa$, and $abaabaa$. In recent years a great deal of work has been done on estimating the maximum number of runs that can occur in a word of length $n$, and a number of associated problems have also been investigated. Recently, Glen and Simpson [The total run length of a word, to appear in *Theoret. Comput. Sci.*, http://arxiv.org/pdf/1301.6568v1.pdf] considered a new variation on the theme by introducing and studying the so-called *total run length* (TRL) of a word, which is defined to be the sum of the lengths of the runs in the word. For example, the word $ababaabaa$ (given above) has total run length $2 \cdot 2 + 5 + 7 = 16$.

It is natural to wonder what is the *maximum* TRL over all words of length $n$. In this direction, we showed in the paper mentioned above that, for all $n$,

$$\frac{n^2}{8} < \tau(n) < \frac{47}{72} n^2 + 2n$$

where $\tau(n)$ denotes the maximum TRL over all words of length $n$. The quadratic upper bound is probably far from best when $n$ is large. From Table 1 below, we suspect that the maximum value of $\tau(n)/n^2$ occurs when $n = 2$ and it would seem that $\lim_{n \to \infty} \tau(n)/n^2$ exists, but we have not yet been able to prove it.
Table 1: Values of $\tau(n)$ for small $n$, assuming these values are attained by binary words.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\tau(n)$</th>
<th>$\tau(n)/n^2$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$a$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.5</td>
<td>$aa$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.333</td>
<td>$aaa$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.250</td>
<td>$aaaa$</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>0.240</td>
<td>$aabab$</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>0.278</td>
<td>$aabaab$</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>0.245</td>
<td>$aabaabb$</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>0.250</td>
<td>$abbaabb$</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
<td>0.235</td>
<td>$abaaabaab$</td>
</tr>
<tr>
<td>10</td>
<td>29</td>
<td>0.290</td>
<td>$aababaabab$</td>
</tr>
<tr>
<td>11</td>
<td>32</td>
<td>0.264</td>
<td>$abaababaaba$</td>
</tr>
<tr>
<td>12</td>
<td>37</td>
<td>0.257</td>
<td>$abaababaabab$</td>
</tr>
<tr>
<td>13</td>
<td>42</td>
<td>0.249</td>
<td>$ababbababbaba$</td>
</tr>
<tr>
<td>14</td>
<td>47</td>
<td>0.240</td>
<td>$aaabaabaabaab$</td>
</tr>
<tr>
<td>15</td>
<td>53</td>
<td>0.236</td>
<td>$abaababaababa$</td>
</tr>
<tr>
<td>16</td>
<td>60</td>
<td>0.234</td>
<td>$aabaababaabaab$</td>
</tr>
<tr>
<td>17</td>
<td>70</td>
<td>0.242</td>
<td>$abaababaabaaba$</td>
</tr>
<tr>
<td>18</td>
<td>73</td>
<td>0.225</td>
<td>$aabaababaabaababa$</td>
</tr>
<tr>
<td>19</td>
<td>80</td>
<td>0.222</td>
<td>$abaababaabaabaaba$</td>
</tr>
<tr>
<td>20</td>
<td>85</td>
<td>0.212</td>
<td>$abaababaabaabababab$</td>
</tr>
<tr>
<td>21</td>
<td>92</td>
<td>0.209</td>
<td>$abaabaababaabaababa$</td>
</tr>
<tr>
<td>22</td>
<td>99</td>
<td>0.205</td>
<td>$abaabaababaababaababa$</td>
</tr>
</tbody>
</table>

We also do not know whether binary words are best, though this seems likely.

Note that if a binary word is optimal with respect to TRL, then so is its reverse and its complement (formed by interchanging the letters $a$ and $b$). In most cases the binary words attaining the values in Table 1 are unique up to reversal and complementation. Also note that, in many cases, binary words having maximal TRL are palindromes.

- Find a closed formula (or at least improve on the known bounds) for $\tau(n)$.
- Find a characterization or construction for (binary) words having maximum TRL.

6 Enumeration of rich words — Amy Glen

of length \(|w|\) contains at most \(|w| + 1\) distinct palindromes (including the empty word). Moreover, a word \(w\) contains exactly \(|w| + 1\) distinct palindromes if and only if the longest palindromic suffix of any prefix \(p\) of \(w\) occurs exactly once in \(p\). Such words are “rich” in palindromes in the sense that they contain the maximum number of different palindromic factors. Accordingly, we say that a finite word \(w\) is rich if it contains exactly \(|w| + 1\) distinct palindromes. Naturally, an infinite word is rich if all of its factors are rich. Rich words were first introduced and studied in 2009 by Glen, Justin, Widmer, and Zamboni [Palindromic richness, European J. Combin. 30 (2009) 510–531] and since then there has been much work done on these words (sometimes called full words) by various authors. However, there still remains many open questions concerning this class of words.

**Open Problem** Find a closed formula (or an asymptotic estimate) for the number of rich words of length \(n\) over a \(k\)-letter alphabet.

### 7 Decidability of divisibility in automata — Jeffrey Shallit

Let \(k \geq 2\) be an integer and let \(\Sigma_k = \{0, 1, \ldots, k - 1\}\).

Let \(M\) be a deterministic finite automaton accepting some words over the alphabet \(\Sigma_k \times \Sigma_k\). Each such word represents a pair of integers \((m, n)\) in base \(k\). Let \(S(M)\) be the set of pairs of integers accepted by \(M\).

**Question.** Is the following problem decidable? Given a DFA \(M\) and its associated \(S(M)\), does there exist an \((m, n) \in S(M)\) such that \(n \mid m\)?

If “does there exist an” is replaced by “is it true that for all” then it is known that the problem is decidable. For more information, see [http://arxiv.org/pdf/1110.2382.pdf](http://arxiv.org/pdf/1110.2382.pdf).

### 8 Partial word representation — Francine Blanchet-Sadri

Given a partial word \(w\), we let \(\text{sub}_w(n)\) denote the set of subwords of \(w\) of length \(n\), i.e., the words that are compatible with factors of \(w\) of length \(n\). We call a set \(S\) of words representable if \(S = \text{sub}_w(n)\) for some integer \(n\) and partial word \(w\). Using a graph-theoretical approach, the problem of whether a given set is representable was shown to be decidable in polynomial time. I am interested in the following two problems:

1. Characterize the sets of words that are representable.
2. Characterize minimal representing words (can they be constructed efficiently?).
9 Palindrome defect conjecture — Michelangelo Bucci

It is well known that the maximum number $P(w)$ of distinct palindromic factors of a finite word $w$ is $|w|+1$ [Droubay, Justin, and Pirillo, Episturmian words and some constructions of de Luca and Rauzy, *Theoret. Comput. Sci.* 255 (2001), 539–553]. The palindromic defect is, in a way, a measure of how far a word is from having the maximum number of palindromic factors (i.e., from being "rich" or "full" in palindromes) and it is defined as $|w|+1−P(w)$ for a finite word and, for infinite words, as the supremum of the palindromic defect of the finite factors.

In 2008, Blondin Massé, Brlek, Garon, and Labbé [Pure Math. Appl., 19 2–3 (2008) 39–52, http://www.mat.unisi.it/newsito/puma/public_html/19_2_3/4.pdf] conjectured that the fixed point of a primitive morphism that is not periodic either has palindromic defect $0$ or infinite. There is a counterexample known for the three-letter alphabet, found last year in Banff by Elise Vaslet and myself: $a \rightarrow aabcacba$, $b \rightarrow aa$, $c \rightarrow a$. However, the binary case and the case of injective morphisms are still open. So, is the following true?

**Conjecture.** The fixed point of a binary primitive morphism that is not periodic either has palindromic defect $0$ or infinite.

10 Class P conjecture — Michelangelo Bucci

A morphism $\varphi$ is said to be in class $P$ if there is a palindrome $p$ such that for each $a \in \Sigma$ there is a palindrome $q_a$ such that $\varphi(a) = pq_a$. See Hof, Knill, and Simon, Singular continuous spectrum for palindromic Schrödinger operators, *Commun. Math. Phys.* 174 (1995), 149–159.

**Conjecture.** Let $w$ be the fixed point of a primitive morphism. Let $\text{Pal}(w)$ denote the set of palindromic factors of $w$. Then $|\text{Pal}(w)| = \infty$ if and only if there is a morphism $\varphi$ such that $\varphi(w) = w$ and $\varphi$ is conjugate to a class $P$ morphism.


11 Open questions related to (generalized) rich words — Štepán Starosta

In two papers by E. Pelantová and myself [Languages invariant under more symmetries: overlapping factors versus palindromic richness; to appear in *Discrete Math.*], available at

The new notions lead to many open questions:

1. What are examples of (classes of) $G$-rich words? So far, the generalized Thue-Morse words were proven to be $G$-rich [Š. Starosta, Generalized Thue-Morse words and palindromic richness, Kybernetika, 48 (2012), 361–370].

2. Given a group $G$ generated by involutive antimorphisms, is there a $G$-rich word?


4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its set of factors has either $G$-defect 0 or $+\infty$; see A. Blondin Massé, S. Brlek, A. Garon, and S. Labbé, Combinatorial properties of $f$-palindromes in the Thue-Morse sequence, Pure Math. Appl., 19 (2008), 39–52. This conjecture is still open for the classical notion of palindromic defect.

5. How many $G$-rich words of length $n$ exist?

12 Avoidability under permutations — Dirk Nowotka

Let $f^i(x)f^j(x)f^k(x)f^\ell(x)$, for some natural numbers numbers $i, j, k, \ell$ be a pattern where $f$ is a variable for a morphic (resp., an anti-morphic) permutation on an alphabet of size $n$ (that is, the pattern occurs, if there is a factor $x$ and a permutation $f$ on the alphabet such that . . .).

Question. When is $f^i(x)f^j(x)f^k(x)f^\ell(x)$ avoidable?

The answer is known for $f^i(x)f^j(x)f^k(x)$, where we have the result that such a pattern can be avoidable for one alphabet size and become unavoidable for a larger one (which is unusual)
13 Text search under involution — Florin Manea

**Question.** Given a text $T$, a word $P$, and an antimorphic involution $f$, can we find all occurrences of $P$ in $T$ under $f$, where a match is also counted for all those $P'$ that result from $P$ after $f$ is applied to some non-overlapping factors of $P$?

The most efficient known algorithms run in $O(|P||T|)$ time and $O(|P|)$ space, in the worst case, and can be adapted to run in linear average time for larger alphabets. Can we get linear average time for all alphabets? Can we solve the problem in less time than $O(|P||T|)$ in the worst case? [Grozea, C., Manea, F., Müller, M. and Nowotka, D., String matching with involutions, *Unconventional Computation and Natural Computation (UCNC)*, LNCS 7445, 2012, pp. 106–117; and the references therein].

14 Parikh matrices — Mike Müller

Parikh matrices are natural generalizations of Parikh vectors, in which not only the occurrences of letters, but also the occurrences of some specific scattered factors of a word are counted. This concept was thoroughly analyzed from the combinatorial and language theoretical point of view, but the most basic algorithmic/complexity problem is still open:

**Question.** How hard is it to decide, for a given matrix $A$, whether there exists a word $w$ whose Parikh matrix $A$?

The problem can be solved efficiently for the case of binary alphabets, but seems to become hard for the case of alphabets with more than three letters (where only a PSPACE-upper bound and a pseudo-polynomial algorithm are known). See also [http://fsl.cs.uiuc.edu/index.php/Parikh_Matrices](http://fsl.cs.uiuc.edu/index.php/Parikh_Matrices).

15 $k$-abelian complexity of $(k-1)$-abelian periodic words — Juhani Karhumäki

(With A. Saarela and L. Zamboni).

The words $u$ and $v$ are $k$-abelian equivalent if $|u|_x = |v|_x$ for all words $x$ of length at most $k$. An infinite word $w$ is $k$-abelian periodic if $w = u_1u_2\cdots$, where $u_1, u_2, \ldots$ are $k$-abelian equivalent. The $k$-abelian complexity of a word $w$ is the function that maps a number $n$ to the number of $k$-abelian equivalence classes of factors of $w$ of length $n$. 
If a word is periodic on a \((k-1)\)-abelian level, how complex can it be on a \(k\)-abelian level? This question is formalized in the following open problem.

**Open Problem.** Let \(k \geq 2\) and let \(w\) be a infinite \((k-1)\)-abelian periodic word. How high can the \(k\)-abelian complexity of \(w\) be?

It is known that

- The maximal \(k\)-abelian complexity of all words is \(\Theta(n^{m^{k-1}})\), where \(m\) is the size of the alphabet [Juhani Karhumäki, Aleksi Saarela, and Luca Zamboni, On a generalization of Abelian equivalence and complexity of infinite words. Submitted; preprint at \text{http://arxiv.org/abs/1301.5104}].

- The image of a binary word with linear abelian complexity under the morphism defined by
  
  \[
  0 \mapsto 0^k10^{k-2}, \quad 1 \mapsto 0^{k-1}10^{k-1}
  \]
  
  is \((k-1)\)-abelian periodic but has linear \(k\)-abelian complexity [Juhani Karhumäki, Aleksi Saarela, and Luca Zamboni, Variations of the Morse-Hedlund theorem for \(k\)-Abelian equivalence, submitted. Preprint at \text{http://arxiv.org/abs/1302.3783}].

The condition of being \((k-1)\)-abelian periodic could be replaced with the weaker condition of having bounded \((k-1)\)-abelian complexity. Then the following related problem could be seen as the “limit” as \(k \to \infty\).

**Problem.** Let \(w\) have bounded \(k\)-abelian complexity for every finite \(k\). How large can the factor complexity of \(w\) be?

It is known that Sturmian words have bounded \(k\)-abelian complexity for every finite \(k\) and their factor complexity is \(n+1\).

## 16 \(k\)-abelian avoidability — Aleksi Saarela

(with M. Huova, J. Karhumäki and R. Mercaș)

The words \(u\) and \(v\) are \(k\)-abelian equivalent if \(|u|_x = |v|_x\) for all words \(x\) of length at most \(k\), where \(|u|_x\) denotes the number of (possibly overlapping) occurrences of \(x\) in \(u\). If \(u_1, \ldots, u_n\) are \(k\)-abelian equivalent, then \(u_1 \ldots u_n\) is an \(k\)-abelian \(n\)th power.

Like avoidability of ordinary powers and avoidability of abelian powers, avoidability of \(k\)-abelian powers can be studied. The interesting cases are cubes on a binary alphabet and squares on a ternary alphabet.

**Problem.** Are \(2\)-abelian cubes avoidable on a binary alphabet?

It is known that
Abelian cubes are not avoidable on a binary alphabet.

The image of Dekking’s ternary abelian cube-free word under the morphism defined by

\[ 0 \mapsto 01010, \quad 1 \mapsto 0110010, \quad 2 \mapsto 0110110, \]

is 3-abelian cube-free [Robert Mercaš and Aleksi Saarela, 3-abelian cubes are avoidable on binary alphabets, *Proc. 17th DLT*, to appear].


**Problem.** For which \( k \) are \( k \)-abelian squares avoidable on a ternary alphabet?

It is known that

- The longest 2-abelian square-free ternary word has length 537 [Mari Huova, Juhani Karhumäki, and Aleksi Saarela, Problems in between words and abelian words: \( k \)-abelian avoidability, *Theoret. Comput. Sci.*, **454** (2012), 172–177].

- Every pure morphic ternary word contains \( k \)-abelian squares for every \( k \) [Mari Huova and Juhani Karhumäki, On the unavoidability of \( k \)-abelian squares in pure morphic words. *J. Integer Seq.*, **16** (2013), Article 13.2.9.]

- 64-abelian squares are avoidable on a ternary alphabet [Mari Huova, Existence of an infinite ternary 64-abelian square-free word, Submitted].

### 17 \( p \)-automatic sequences: from equation to automaton — Eric Rowland

*Given an algebraic description of a \( p \)-automatic sequence, how should one compute an automaton for the sequence?*

Let \( \mathbb{F}_q \) be a finite field of characteristic \( p \). Christol et al. [Christol, Kamae, Mendès France, and Rauzy, Suites algébriques, automates et substitutions, *Bull. Soc. Math. France* **108** (1980) 401–419]. showed that a sequence \( s(n)_{n \geq 0} \) of elements in \( \mathbb{F}_q \) is \( p \)-automatic if and only if the formal power series \( \sum_{n \geq 0} s(n)x^n \) is algebraic over \( \mathbb{F}_q(x) \). Given a \( p \)-automatic sequence, one can compute a polynomial \( P(x, y) \in \mathbb{F}_q[x, y] \) such that \( P(x, \sum_{n \geq 0} s(n)x^n) = 0 \) by computing the null space of a matrix whose entries are univariate polynomials of degrees at most \( p^{(\text{size of the } p\text{-kernel})+1} \). This algorithm is reasonably fast in practice.

However, given \( P(x, y) \in \mathbb{F}_q[x, y] \), what algorithm should one use to compute a finite automaton for a sequence \( s(n)_{n \geq 0} \) such that \( P(x, \sum_{n \geq 0} s(n)x^n) = 0 \)? The existence proof
given in Theorem 12.2.5 of Allouche & Shallit, *Automatic Sequences*, uses Ore’s lemma to first put \( P(x, y) \) into the form \( \sum_{i=0}^{\deg_y P(x,y)} c_i(x)y^i \).

Constructive versions of Ore’s lemma are given in Lemma 3.1.3 of Pytheas Fogg [Substitutions in Dynamics, Arithmetics, and Combinatorics (Lecture Notes in Mathematics, Vol. 1794), Springer, 2002] and Lemma 8.1 of Adamczewski & Bell [On vanishing coefficients of algebraic power series over fields of positive characteristic, *Inventiones Math.* 187 (2012) 343–393], but in practice the \( y \)-degree \( \deg_y P(x,y) \) of the Ore polynomial in \( y \) is prohibitively large for all but small-degree polynomials. This is because, even when \( \deg_y P(x,y) \) is small and the \( p \)-kernel of \( s(n)_{n \geq 0} \) is small, the intermediate expressions are large. A technique of Denef & Lipshitz [Algebraic power series and diagonals, *J. Number Theory* 26 (1987) 46–67; Remark 6.6] based on work of Christol [´El´ ements analytiques uniformes et multiformes, Séminaire Delange–Pisot–Poitou (Théorie des nombres) 15 (1973/1974) no. 6] seems to be substantially faster in some cases but requires the polynomial to be in a certain form. What algorithm should one use in general?

### 18 Repetitions in Sturmian words — Boris Adamczewski

Let \( \mathcal{A} \) be a finite set. As usual \(|w|\) denotes the length of the word \( w \) and \( w^k = \underbrace{ww \cdots w}_k \). More generally, for any positive real number \( \beta \), let \( w^{\beta} \) denote the word \( w^\lfloor \beta \rfloor w' \), where \( w' \) is the prefix of \( w \) of length \( \lceil (\beta - \lfloor \beta \rfloor) |w| \rceil \).

We now consider a natural measure of repetitions occurring in infinite words. It was formally introduced by Adamczewski & Bugeaud [Dynamics for \( \beta \)-shifts and Diophantine approximation, *Ergod. Th. & Dynam. Sys.* 27 (2007), 1695–1710; also see Adamczewski & Cassaigne, Diophantine properties of real numbers generated by finite automata, *Compositio Math.* 142 (2006), 1351–1372].

The *Diophantine exponent* of an infinite word \( a \) is defined as the supremum of the real numbers \( \rho \) for which there exist arbitrarily long prefixes of \( a \) that can be factorized as \( UV^\gamma \), where \( U \) and \( V \) are two finite words (\( U \) possibly empty) and \( \gamma \) is a real number such that

\[
\frac{|UV^\gamma|}{|UV|} \geq \rho.
\]

Of course, we always have the trivial relation

\[1 \leq \text{dio}(a) \leq +\infty.\]

The problem is to understand the diophantine exponent of Sturmian words. I suggest considering the following two problems.
**Problem.** Let $s$ be a Sturmian word, say with slope $\alpha$ and intercept $\rho$. Give a formula for $\text{dio}(s)$ in terms of $\alpha$ and $\rho$ (for instance, depending on the continued fraction expansion of $\alpha$ and the Ostrowski expansion of $\rho$).

**Conjecture.** Let $s$ be a Sturmian word. Then

$$\text{dio}(s) \geq \frac{3 + \sqrt{5}}{2}.$$ 

19 **Iterated hairpin completion — Steffen Kopecki**

The iterated hairpin completion of a word, an operation inspired by the hairpin formation in DNA biochemistry, is always a context-sensitive language and for some words it is known to be not context-free. Two problems naturally emerge:

1. Characterize the class of words whose iterated hairpin completion is regular (by decidable properties).

2. Does a word exist whose iterated hairpin completion is context-free but not regular?

Both problems have been solved for the special class of non-crossing words by Kari, Kopecki, and Seki; namely, the iterated hairpin completion of a word is either regular or not context-free, and it is decidable which one applies (by comparing certain prefixes and suffixes of the word). Even though both problems are still unresolved for the general case, some of the structural properties of hairpin completions of non-crossing words can be extended to the general case.

20 **Bifix codes on smooth words — Michael Rao**

Let $\Delta : \Sigma^* \to (\mathbb{N}^+)^*$ be the run-length-encoding function; that is, $\Delta(x_1^{k_1}x_2^{k_2} \cdots) = k_1k_2 \cdots$, where $k_i > 0$ and $x_i \neq x_{i+1}$ for every $i$.


$$\kappa = 12211212212 \cdots$$

is a fixed point of $\Delta$ over the alphabet $\{1, 2\}$.

Let $\delta(w)$ be $\Delta(w)$, without the first and/or the last letter if it is 1, e.g., $\delta(1) = \delta(21) = \epsilon$, $\delta(2212) = 21$. 
A finite word \( w \in \{1, 2\}^* \) is smooth if either \( w \) is empty, or \( \delta(w) \in \{1, 2\}^* \) and \( \delta(w) \) is smooth. Let \( C^\infty \) be the set of finite smooth words. It is conjectured that \( C^\infty \) is exactly the set of factors of \( \kappa \).


\[
\pi(w) = \frac{1}{s(w) \cdot 3^h(w)},
\]
where \( s(\epsilon) = 1, s(1) = s(2) = 2, s(w) = s(\delta(w)), h(1) = h(2) = h(\epsilon) = 0, \) and \( h(w) = 1 + h(\delta(w)) \).

A code \( W \subseteq \Sigma^* \) is a prefix (resp., suffix) code if for every \( u, v \in W \), \( u \neq v \), \( u \) is not a prefix (resp., suffix) of \( v \). A code is a bifix code if it is a prefix code and a suffix code. A code is \( L \)-maximal, for a language \( L \subseteq \Sigma^* \), if \( W \subseteq L^* \) and there is no code \( W' \subseteq L \) such that \( W \subset W' \).

A parse of a word \( w \) with respect to \( W \) is a triple \((v, x, u)\) such that \( w = vxu \), \( x \in W^* \), \( v \in \Sigma^* \setminus \Sigma^* W \) and \( u \in \Sigma^* \setminus W \Sigma^* \). Let \( d_W(w) \) be the number of parses of \( w \) with respect to \( W \).

**Conjecture.** Let \( W \) be a \( C^\infty \)-maximal bifix code. Then \( d(W) = \sum_{w \in W} |w| \cdot \pi(w) \) is an integer (if it is finite), and is equal to the maximal number of parses of a smooth word, that is:

\[
d(W) = \max_{w \in C^\infty} d_W(w).
\]

**Remark:** if the Universal Gluing Conjecture below is true, then the previous conjecture is true by a result in [J. Berstel, C. De Felice, D. Perrin, C. Reutenauer, G. Rindone, Bifix codes and Sturmian words, J. Algebra, 369 (2012), 146–202].

**Universal Gluing Conjecture.** For every \( u, v \in C^\infty \), there is a \( w \in C^\infty \) such that \( uwv \in C^\infty \).

**Question.** Is there an infinite \( C^\infty \)-maximal bifix code \( W \) with \( d(W) < \infty \)?

Let \( W \) be a \( C^\infty \)-maximal bifix code. Let

\[
W' = \{ \Delta(u) : u \in W^+, \Delta(u) \in C^\infty, u[1] = 1 \text{ and } u[|u|] = 2 \}.
\]

Let \( W'' \) be the set of minimal elements in \( W' \) under the prefix order.

**Conjecture.** \( W'' \) is a \( C^\infty \)-maximal bifix code with \( d(W'') = 2 \cdot d(W) \).

21 Avoidability problems — James Currie

1. Is there a word which is 6-avoidable (i.e., avoidable over a 6-letter alphabet) but not 5-avoidable? Is there always a word which is \((k + 1)\)-avoidable but not \(k\)-avoidable? What can be said for the Abelian version?

2. Fix a ternary alphabet \(S\). Suppose \(x\) (resp., \(y\)) is a prefix (resp., suffix) of infinitely many squarefree words over \(S\). Is there a squarefree word over \(S\) with prefix \(x\) and suffix \(y\)?

3. What is the complexity of deciding whether a given pattern \(p\) is avoidable?