Correction and details on Kunen-Miller proposition in Cohen model

I realized I made a simple error in discussing the proof and that I wanted to explain in more detail how to use make use of the fact that the elementary submodel we chose satisfied that $M^\omega \subset M$.

First, in case this is new to you, here is how we get such models.

Start with any countable model $M_0 \prec H(\theta)$ (e.g. $\theta = \mathfrak{c}^+$. Recursively choose, for $\alpha \in \omega_1$, a model $M_\alpha \prec H(\theta)$ so that, by induction, $|M_\alpha| \leq \mathfrak{c}$, and the size $\mathfrak{c}$ set $[\bigcup\{M_\beta : \beta < \alpha\}]^\omega$ is an element and contained in M_α. A theorem of Tarski implies that $M_{\omega_1} = \bigcup\{M_\alpha : \alpha \in \omega_1\}$ is an elementary submodel of $H(\theta)$. Easy to check that $[M_{\omega_1}]^\omega \subset M_{\omega_1}$ and that $|M_{\omega_1}| = \mathfrak{c}$.

The mistake I made was after choosing $M \prec H(\theta)$ with $|M| = \omega_1$ and describing λ as $M \cap \omega_2$, I acted as though $M \cap \omega_3$ is also equal to λ. Well, it is not, so instead, let G be $Fn(\omega_3, 2)$-generic. Let $I = M \cap \omega_3$, and set $G_M = G \cap M$ which is $Fn(I, 2)$-generic, and believe (just the way we did with $Fn(\lambda, 2)$) that $V[G]$ is obtained by forcing over $V[G_M]$ with the poset $Fn(\omega_3 \setminus I, 2)$.

The other very important thing is how to fully use the elementarity of M. As I said, since $M^\omega \subset M$, we know that $V[G_M]$ and $M[G_M]$ have the same countable sets of ordinals – hence the same subsets of ω. Also, what is $M[G_M]$? Well, of course, it is just the valuations of the names that are members of M. A straightforward exercise though, is that $M[G_M]$ is an elementary submodel of $H(\theta)[G]$ (the full G).

Theorem 1. *Let G be $Fn(\omega_3, 2)$-generic over $V \models CH$. Let $\mathcal{A} = \{\dot{a}_\alpha : \alpha \in \omega_2\}$ be names of subsets of ω. Let $\mathcal{A} \in M \prec H(\theta)$ such that $M^\omega \subset M$ and $|M| = \omega_1$. The following are true in $V[G]$:*

1. *(Kunen) the family $val_G(\mathcal{A}) = \{val_G(\dot{a}_\alpha) : \alpha \in \omega_2\}$ is not a mod finite chain*
2. *(Miller) the family $val_G(\mathcal{A})$ is not maximal almost disjoint.*

Proof. (1) We let $\mathcal{F}$ denote the collection of $\dot{b}$ which are forced to be subsets of ω which mod finite contain every member of $\mathcal{A}$. We now pass to the model $V[G_M]$ and we check that

for each $\alpha \in \lambda$, $\dot{a}_\alpha \in M$ and so $a_\alpha = val_{G_M}(\dot{a}_\alpha)$ is in $V[G_M]$

and, for each $\alpha \in \omega_2 \setminus \lambda$, we may assume that $\dot{a}_\alpha$ is not in M because we are assuming that $V[G]$ sees this as an ω_2-chain and $V[G_M]$ is a model of CH and so $\dot{a}_\lambda$ (for example) can not be a $Fn(M \cap \omega_3, 2)$-name.

Using that $M^\omega = M$, check that, in $V[G_M]$, for all $c \subset \omega$ such that $a_\alpha \subset^* c$ for all α, there is a $\dot{b} \in \mathcal{F} \cap M$ such that $val_{G_M}(\dot{b}) = c$.

Here's something we should have talked about: we now know that the name $\dot{a}_\lambda$ can be thought of as a $Fn(\omega_3 \setminus I, 2)$-name. But also, remember that we can assume that $\dot{a}_\lambda$ is actually a "nice" name (for each n, there is an antichain A_n so that $\dot{a}_\lambda$ is just the union of the collection $\check{n} \times A_n$). This means that there is a countable subset J of $\omega_3 \setminus I$ such that $\dot{a}_\lambda$ is simply an $Fn(J, 2)$-name. The point being that $Fn(J, 2)$ is countable.

For each $\alpha \in \lambda$, there is a condition $p_\alpha \in Fn(J, 2)$ and an integer n_α so that $p_\alpha \Vdash a_\alpha \setminus \dot{a}_\lambda \subset n_\alpha$. Choose a cofinal set $\Gamma \subset \lambda$ and a single pair p, n so that $p_\alpha = p$ and $n_\alpha = n$ for all $\alpha \in \Gamma$.

But this is bad: notice that we have that p forces that $\dot{a}_\lambda$ contains the set $Y = \bigcup\{a_\alpha : \alpha \in \Gamma\}$ which is in the model $V[G_M]$. Since Y contains mod finite all the sets $\{a_\beta : \beta < \lambda\}$, this means that Y is in the family $\{val_{G_M}(\dot{b}) : \dot{b} \in \mathcal{F} \cap M\}$. This contradicts that p forces that $\dot{a}_\lambda$ is *properly* contained in every member of $\mathcal{F}$.

(2) Now suppose that some $p \in M$ forces that $\mathcal{A}$ is a mad family. Observe that there is an infinite $J \subset \omega_3$ such that, for all $\alpha \in \omega_2$, we have that $\dot{a}_\alpha$ is an $Fn(\omega_3 \setminus J, 2)$-name. We may assume that $J \in M$. Now we pass to the model $V[G_M]$ and we more carefully examine the almost disjoint family $\{a_\alpha : \alpha \in \lambda\}$. We again factor the forcing, but not with any elementary submodels, we just take the generic $G' = G_M \cap Fn(M \setminus J, 2)$. The family $\{a_\alpha : \alpha \in \lambda\}$ is unaffected. But the main thing is, is that it has the property that if we then force with $Fn(J, 2)$ it remains maximal because it is maximal in the model $V[G_M]$. But!! $\dot{a}_\lambda$ is a Cohen name using a different index set. There is an isomorphism from the "support" of $\dot{a}_\lambda$ into J which sends $\dot{a}_\lambda$ to a $Fn(J, 2)$-name $\dot{b}$ which is forced to be almost disjoint from each a_α $(\alpha < \lambda)$.

$\square$