# Intersection bounds for nodal sets of planar Neumann eigenfunctions with interior analytic curves

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Weyl law at 100 (Fields Institute)

September 19-21, 2012

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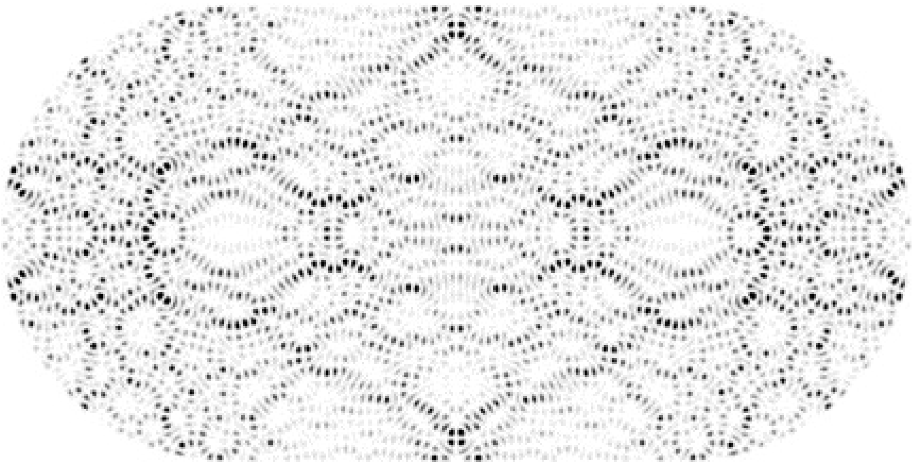
- **QUESTION:** As $\lambda \to \infty$, how many nodal lines (components of the nodal set) intersect a fixed interior real analytic curve $H$?

## Probability density plot of an eigenstate of a Bunimovich stadium (courtesy of M.F. Andersen, A. Kaplan, T. Grünzweig and N. Davidson)



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## Theorem

*Theorem (Zelditch-T (2009)) Let $H$ be a real analytic interior curve that is good.Then, there is a constant $C_{\Omega,H} > 0$ such that for all Neumann eigenfunctions $\phi_\lambda$,*
$$n(\lambda, H) \leq C_{\Omega,H}\lambda.$$
*When $H = \partial\Omega$,*
$$n(\lambda, \partial\Omega) \leq C_\Omega \lambda.$$

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- Easy to see that not all interior curves are good. For example, the Neumann eigenfunctions for the disc in polar variables $(r,\theta) \in (0,1] \times [0,2\pi]$ are

$$\phi_{m,n}(r,\theta) = C_{m,n} \cos m\theta J_m(j'_{m,n} r) \ (C_{m,n} \sin m\theta J_m(j'_{m,n} r)).$$

Here, $J_m$ is the $m$-th integral Bessel function and $j'_{m,n}$ is the $m$-th critical point of $J_m$. The eigenvalues are $\lambda^2_{m,n} = (j'_{m,n})^2$.

# Positive results known

- Fix $m \in Z^+$ and consider

$$H_m = \{(r, \theta); \theta = \frac{2\pi k}{m}; \; k = 0, ..., m-1\}.$$

Then, clearly for all $n$, $\quad \phi_{m,n}|_{H_m} = 0$, and so $H_m$ is not good.

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- When $(M^n, g)$ is a flat torus with $n = 2, 3$, and $H \subset M$ has strictly positive curvature (Bourgain-Rudnick(2010))

$$\int_H |\phi_\lambda|^2 d\sigma \approx 1.$$

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- Closed horocycles $H$ in finite-volume hyperbolic surfaces are good (Jung(2011)) and so the $O_H(\lambda)$ intersection bound holds.

# Theorem 1 [El-Hajj - T (2012)]

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*Let* $\Omega$ *be a bounded, piecewise-analytic domain and* $H \subset \mathring{\Omega}$ *an interior,* $C^\omega$ *curve with restriction map* $\gamma_H : C^0(\Omega) \to C^0(H)$.

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*Let* $H^{\mathbb{C}}_{\epsilon_0}$ *denote the complex radius* $\epsilon_0 > 0$ *Grauert tube containing* $H$ *as its totally real submanifold and* $(\gamma_H \phi_\lambda)^{\mathbb{C}}$ *be the holomorphic continuation of* $\gamma_H \phi_\lambda$ *to* $H^{\mathbb{C}}_{\epsilon_0}$.

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*Suppose the curve H satisfies the* revised goodness condition

$$\sup_{z \in H_{\epsilon_0}^{\mathbb{C}}} |(\gamma_H \phi_\lambda)^{\mathbb{C}}(z)| \geq e^{-C_0 \lambda} \qquad \text{for some } C_0 > 0. \quad (*)$$

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*Then, there is a constant $C_{\Omega,H} > 0$ such that for all $\lambda \leq \lambda_0$,*

$$n(\lambda, H) \leq C_{\Omega,H} \lambda.$$

# Key tool: potential layer

- An important point is that $(*)$ can be verified using $T^*T$-type operator bounds for the holomorphic continuation to $H^{\mathbb{C}}_{\epsilon_0}$ of the potential layer operator $N(\lambda) : C^\infty(\partial\Omega) \to C^\infty(H)$

$$N(\lambda)(x, y) = \int_{\partial\Omega} \partial_{\nu_y} G_0(x, y, \lambda)\, d\sigma(y),$$

where,

$$G_0(x, y, \lambda) = \frac{i}{4} \mathrm{Ha}_0^{(1)}(\lambda|x - y|).$$

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*Then,*

$$n(h_{j_k}, H) = \mathcal{O}_{H,\Omega}(h_{j_k}^{-1}).$$

# Theorem 3 [Canzani-T(2012)]

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*Let $(M^2, g)$ be a compact, real-analytic surface with $\partial M = \emptyset$ and ergodic geodesic flow $G^t : S^*M \to S^*M$. Let $H \subset M^2$ be a real-analytic closed curve with strictly-positive geodesic curvature.*

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- When $(M^2, g)$ is quantum uniquely ergodic (QUE), the intersection bound in Theorem 3 holds for all eigenfunctions. For example, this is the case when $M = \Gamma / \mathbf{H}$ is arithmetic (Lindenstrauss (2006)).
- One can formulate a more general version of Theorem 2 in terms of defect measures which need not be ergodic (examples?)

# Outline of the proof of Theorem 1

- We want to show that $n(h, H) = \mathcal{O}(h^{-1})$ under the assumption that $sup_{z \in H^{\mathbb{C}}_{\epsilon_0}} |\phi_h^{H,\mathbb{C}}(z)| \geq e^{-C/h}$ for some $C > 0$.

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- Let $q : [-\pi, \pi] \to H$ be a $C^{\omega}$-parametrization of a closed curve $H$ with $|q'(t)| \neq 0$ and $q(t + 2\pi) = q(t)$.

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- Consider the eigenfunction restriction,

$$u_h^H(t) = \phi_h(q(t)),\ t \in [-\pi, \pi]$$

  and complexify $u_h^H$ to a holomorphic function $u_h^{H,\mathbb{C}}(t)$ with $t \in S_{2\epsilon_0}$ where

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- Let $C_{\epsilon_0}\subset S_{2\epsilon_0}$ be a simply-connected domain with $C^{\omega}$ boundary $\partial C_{\epsilon_0}$ containing the interval $[-\pi,\pi]$.

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- Assuming $u_h^{H,\mathbb{C}}(t) \neq 0$ for all $t \in C_{\epsilon_0}$, frequency function method of Han-Lin gives the upper bound

$$n(h,H) \leq C_1 \left( \frac{\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} \right). \tag{2}$$

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- In (2), $L^2_{\epsilon_0} := L^2(\partial C_{\epsilon_0}, d\sigma(t))$ and $\partial_T$ is the unit tangential derivative along $\partial C_{\epsilon_0}$.

# Outline of the proof of Theorem 1

- We $h$-microlocally decompose the right hand side in (2). Let $\chi_R \in C_0^\infty(T^*\partial C_{\epsilon_0})$ with $\chi_R(s,\sigma) = 1$ for $|\sigma| \leq R+1$ and $\chi_R(s,\sigma) = 0$ for $|\sigma| \geq R+2$ with $R > 1$ sufficiently large.

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  Clearly,

$$\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} \le \|\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} + \|\partial_T(1 - Op_h(\chi_R)) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}. \tag{3}$$

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- Since $h\partial_T Op_h(\chi_R) \in Op_h(S^{0,0}(T^*\partial C_{\epsilon_0}))$, by $L^2$-boundedness one estimates the first term on RHS of (3):

$$\frac{\|\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} = h^{-1} \frac{\|h\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} = \mathcal{O}(h^{-1}). \tag{4}$$

# Outline of the proof of Theorem 1

- To estimate the right hand side of (3), we use potential layer formulas combined with a complex contour deformation argument to show that

$$\|h\partial_T(1 - Op_h(\chi_R))u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} = \mathcal{O}(e^{-C_R/h}).$$

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  Here, $C_R \gg R$ as $R \to \infty$.

- Choose the strip $S_{\epsilon_0,\pi} = \{t \in \mathbb{C}; -\pi \leq \Re t \leq \pi, |\Im t| < \epsilon_0\}$ with $S_{\epsilon_0,\pi} \subset \text{Int}(C_{\epsilon_0})$. By Cauchy integral formula, Cauchy-Schwarz and the goodness condition $(*)$,

  $$\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} \geq C \cdot \sup_{t \in S_{\epsilon_0,\pi}} |u_h^{H,\mathbb{C}}(t)| \geq e^{-C_0/h}.$$

# Outline of the proof of Theorem 1

- It follows that

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- By choosing $R$ sufficiently large in the radial frequency cutoff $\chi_R$, we get that $C_R - C_0 \gg R > 0$.

# Outline of proof of Theorem 2

- We consider here the case where $\Omega$ is a bounded, convex planar domain with ergodic billiards and that $(\phi_{h_j})$ is a sequence of QE interior eigenfunctions. We want to show that $\sup_{z \in H^{\mathbb{C}}_{\epsilon_0}} |\phi_h^{H,\mathbb{C}}(z)| \geq e^{-C/h}$ in the case where $H \subset \Omega$ is an interior curve with $\kappa_H > 0$. We do this by proving some weighted-$L^2$ lower bounds.

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- Let $H^{\mathbb{C}}(\epsilon_0)$ be a complex Grauert tube of radius $\epsilon_0 > 0$ with totally-real part $H$ and $\zeta_{\epsilon_0} \in C^\infty(H^{\mathbb{C}}(\epsilon_0); [0,1])$ be a cutoff on the Grauert tube equal to $1$ on the annulus $H^{\mathbb{C}}(\epsilon_0/2) - H^{\mathbb{C}}(\epsilon_0/3)$ and vanishing outside.

# Outline of proof of Theorem 2

- The main technical part of the proof of Theorem 2 consists of showing that under the non-vanishing curvature condition on $H$ and for $\epsilon_0 > 0$ small, there is an order-zero semiclassical pseudodifferential operator

$$P(h) \in Op_h(S^{0,0}(T^*\partial\Omega))$$

and a weight function

$$\rho \in C^\omega(\operatorname{supp} \zeta_{\epsilon_0}; \mathbb{R}^+)$$

such that

$$h^{-1/2} \int\int_{\mathbb{C}} e^{-2\rho(t)/h} |u_h^{H,\mathbb{C}}(t)|^2 \, \zeta_{\epsilon_0}(t) \, dt d\bar{t} \sim_{h\to 0^+} \langle P(h)\phi_h^{\partial\Omega}, \phi_h^{\partial\Omega}\rangle. \tag{5}$$

# Outline of proof of Theorem 2

- The potential layer formula gives

$$\phi_h^H = \gamma_H N(h) \phi_h^{\partial\Omega}, \qquad (u_h^H(t) = \phi_h^H(q(t))),$$

with holomorphic continuation

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- Writing the LHS of (5) as a composition, reduced to proving that $P(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ with
$$P(h) = (e^{-\rho/h} \zeta_{\epsilon_0} \gamma_H^{\mathbb{C}} N^{\mathbb{C}}(h))^* \circ (e^{-\rho/h} \zeta_{\epsilon_0} \gamma_H^{\mathbb{C}} N^{\mathbb{C}}(h))$$
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- Here, $N^{\mathbb{C}}(h)$ is holomorphic continuation of the potential layer operator $N(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ and $\gamma_H^{\mathbb{C}} : \Omega_{\epsilon_0}^{\mathbb{C}} \to H_{\epsilon_0}^{\mathbb{C}}$ is restriction.

# Outline of proof of Theorem 2

- The principal symbol $\sigma(P(h))$ satisfies

$$\int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dyd\eta \geq C_{H,\Omega,\epsilon_0} > 0$$

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# Outline of proof of Theorem 2

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- Given a quantum ergodic sequence $(\phi_{h_{j_k}})_{k=1}^{\infty}$, the boundary restrictions $(\phi_{h_{j_k}}^{\partial\Omega})_{k=1}^{\infty}$ are themselves quantum ergodic (Burq,Hassell-Zelditch) in the sense that

$$\langle P(h)\phi_h^{\partial\Omega}, \phi_h^{\partial\Omega}\rangle \sim_{h\to 0^+} \int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dyd\eta. \tag{6}$$

# Outline of proof of Theorem 2

- It follows that

$$h^{-1/2}\int\int_{\mathbb{C}} e^{-2\rho(t)/h}|u_h^{H,\mathbb{C}}(t)|^2\,\zeta_{\epsilon_0}(t)\,dtd\bar{t} \sim_{h\to 0^+} \int_{B^*\partial\Omega}\sigma(P(h))\gamma^{-1}\,dyd\eta = C_{\Omega,H,\epsilon_0} > 0 \tag{7}$$

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$$\sim_{h \to 0^+} \int_{B^* \partial\Omega} \sigma(P(h)) \gamma^{-1} \, dy d\eta = C_{\Omega, H, \epsilon_0} > 0$$

- The lower bound in (7) implies that the revised goodness condition $(*)$ must be satisfied.

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- Main point is to prove that $P(h) : C^\infty(M) \to C^\infty(M)$ with

$$P(h) = (e^{-\rho/h}\zeta_{\epsilon_0}\gamma_H^{\mathbb{C}} W(h)^{\mathbb{C}})^* \circ (e^{-\rho/h}\zeta_{\epsilon_0}\gamma_H^{\mathbb{C}} W(h)^{\mathbb{C}})$$

is $h$-pseudodifferential.

# Questions and Remarks

- **(i)** Derive an asymptotic density of states formula for complex zeros of $\phi_h^{H,\mathbb{C}}$ when $H$ is geodesically curved (in analogy with the case where $H$ is geodesic (Zelditch (2012)).

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- **(ii)** Upper bounds for $n(H, \lambda)$ for more general (non-ergodic) domains when $H$ is curved.

- **(iii)** Polynomial lower bounds for $n(H, \lambda)$ when $H$ is either an interior curve or $H = \partial\Omega$.