The Geometry of Light Transport Theory

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Light Transport Theory
quantum electrodynamics
quantum electrodynamics

\[\text{large number of photons}\]

Maxwell’s equations
quantum electrodynamics

\[\text{large number of photons}\]

Maxwell’s equations

\[\text{short wavelength limit}\
\text{neglect of polarization}\]

generic optical
quantum electrodynamics

Maxwell’s equations

light transport theory
quantum electrodynamics

\[ \text{large number of photons} \]

Maxwell’s equations

\[ \text{short wavelength limit} \]
\[ \text{neglect of polarization} \]

light transport theory

\[ \text{neglect of intensity} \]

generic optic
"Theoretical photometry constitutes a case of ‘arrested development’, and has remained basically unchanged since 1760 while the rest of physics has swept triumphantly ahead. In recent years, however, the increasing needs [. . .] have made the absurdly anti-quated concepts of traditional photometric theory more and more untenable."¹

Current light transport theory

Transport equation

\[ \nabla_x L(x, \omega) = 0 \]
Current light transport theory

Transport equation

\[ \nabla_x L(x, \omega) = 0 \]

radiance
Current light transport theory

Transport equation

\[ \nabla_x L(x, \omega) = 0 \]

radiance
Current light transport theory

Transport equation

\[ \nabla_x L(x, \omega) = 0 \]

subject to

\[ L(x, \bar{\omega}) = \int_{H_x^2} L(x, \omega) \rho_x(\omega, \bar{\omega}) \, d\omega \]
configuration space $Q \subset \mathbb{R}^3$

\[
\frac{\partial}{\partial t} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\varepsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}
\]
\[ f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \]  
\[ = \sum_{i=1}^{n} \langle f(\bar{x}), \tilde{\phi}_i(\bar{x}) \rangle \phi_i(x) \]  
\[ f(\lambda) = \langle f(\bar{x}), k_\lambda(\bar{x}) \rangle = \delta_\lambda(f) \]  
\[ f(x) = \sum_{i=1}^{n} \langle f(\bar{x}), k_\lambda_i(\bar{x}) \rangle \tilde{k}_i(x) \]  
\[ = \sum_{i=1}^{n} f(\lambda_i) \tilde{k}_i(x) \]  
\[ \bar{f}_\kappa = f(x) + \kappa \prod_{i=1}^{n} (x - \lambda_i)^2 \]  
\[ Q^*Q \]  
\[ \eta^* t \ell \dot{\ell} = -\dot{\xi} X H \ell \]  
\[ \eta^* t \ell \dot{\ell} = -\dot{\xi} X H \ell \]  
\[ \eta^* t \ell \dot{\ell} = -\dot{\xi} X H \ell \]  
\[ T^*Q X^H \]  
\[ g^* \dot{\ell} = a d^* H \delta \ell \text{Diff}_{\text{can}}(T^*Q) \]
emagnetic theory

configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

microlocal analysis
(Wigner transform)

phase space $T^*Q$

$$\dot{W}^\epsilon = -\{p^\epsilon, W^\epsilon\}$$
configuration space \( Q \subset \mathbb{R}^3 \)

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right) - \frac{\dot{W}}{\epsilon} = \frac{1}{\epsilon} \left[ p^\epsilon, W^\epsilon \right] + \frac{1}{i} \left\{ p^\epsilon, W^\epsilon \right\} + O(\epsilon)
\]

phase space \( T^*Q \)

\[
-\dot{W}^\epsilon = \frac{1}{\epsilon} \left[ p^\epsilon, W^\epsilon \right] + \frac{1}{i} \left\{ p^\epsilon, W^\epsilon \right\} + O(\epsilon)
\]
configuration space \( Q \subset \mathbb{R}^3 \)

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \epsilon \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right)
\]

phase space \( T^*Q \)

\[
-\dot{W}^\epsilon = \frac{1}{\epsilon} [p^\epsilon, W^\epsilon] + \frac{1}{i} \{p^\epsilon, W^\epsilon\} + O(\epsilon)
\]

\[
\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]
\]

microlocal analysis (Wigner transform)

\( \varepsilon \to 0 \)
configuration space $Q \subset \mathbb{R}^3$

\[
\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\varepsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)
\]

phase space $T^*Q$

\[
-\dot{W}^\varepsilon = \frac{1}{\varepsilon} [p^\varepsilon, W^\varepsilon] + \frac{1}{i} \{p^\varepsilon, W^\varepsilon\} + O(\varepsilon)
\]

$\varepsilon \to 0$

\[
\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]
\]

unpolarized radiation

light transport equation

\[
\dot{\ell} = -\{\ell, H\}
\]
\[ f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \]  
\[ f(\lambda) = \langle f(\bar{x}), k \lambda(\bar{x}) \rangle = \delta_{\lambda}(f) \]  
\[ f(x) = \sum_{i=1}^{n} \langle f(\bar{x}), k \lambda_i(\bar{x}) \rangle \tilde{k}_i(x) \]  
\[ \bar{f}_\kappa = f(x) + \kappa \prod_{i=1}^{n} (x - \lambda_i) \]  

\[ Q^T Q \]  
\[ \eta^* t \ell \dot{\ell} = -\mathcal{X} H \ell \]  
\[ \eta^* t \ell \dot{\ell} = -\mathcal{X} H \ell \]  
\[ T^* Q X^H \]  
\[ g^* \dot{\ell} = a \delta H \delta_{\text{can}} (T^* Q) \]
\[ f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \quad (1) \]
\[ f(\lambda) = \langle f(\bar{x}), k \lambda(\bar{x}) \rangle = \delta \lambda(f) \quad (3) \]
\[ f(x) = \sum_{i=1}^{n} \langle f(\bar{x}), k \lambda_i(\bar{x}) \rangle \tilde{k}_i(x) \quad (4) \]
\[ \bar{f}_\kappa = f(x) + \kappa \prod_{i=1}^{n} (x - \lambda_i) \quad (6) \]
\[ Q^T \quad (7) \]
\[ \eta^* \ell \dot{\ell} = -X H(\ell) \quad (8) \]
\[ \eta^* \ell \dot{\ell} = -X H(\ell) \quad (9) \]
\[ \eta^* \ell \dot{\ell} = -X H(\ell) \quad (10) \]
\[ T^* Q \quad (11) \]
\[ g^* \dot{\ell} = \delta H \quad (12) \]
\[ f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \]  

\[ = \sum_{i=1}^{n} \langle f(\bar{x}), \tilde{\phi}_i(\bar{x}) \rangle \phi_i(x) \]  

\[ f(\lambda) = \langle f(\bar{x}), k\lambda(\bar{x}) \rangle = \delta_\lambda(f) \]  

\[ f(x) = \sum_{i=1}^{n} \langle f(\bar{x}), k\lambda_i(\bar{x}) \rangle \tilde{k}_i(x) \]  

\[ \bar{f}_\kappa = f(x) + \kappa \prod_{i=1}^{n} (x - \lambda_i) \]  

\[ Q^T Q \]  

\[ \eta^* t \ell \dot{\ell} = -X^H \]  

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\[ T^* Q X^H \]  

\[ g^* \dot{\ell} = a d^* \delta H \delta \ell \text{Diff} \]  

\[ T^* Q \]
\[ f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \]  
\[ = \sum_{i=1}^{n} \langle f(\bar{x}), \tilde{\phi}_i(\bar{x}) \rangle \phi_i(x) \]  
\[ f(\lambda) = \langle f(\bar{x}), k \lambda(\bar{x}) \rangle = \delta(\lambda) f \]  
\[ f(x) = \sum_{i=1}^{n} \langle f(\bar{x}), k \lambda_i(\bar{x}) \rangle \tilde{k}_i(x) \]  
\[ \bar{f}\kappa = f(x) + \kappa \prod_{i=1}^{n} (x - \lambda_i) \]  
\[ Q^T Q \]  
\[ \eta^* t \dot{\ell} = -X^* H \ell \]  
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\[ \eta^* t \dot{\ell} = -X^* H \ell \]  
\[ T^* Q X^H \]  
\[ g \dot{\ell} = \delta H \delta \ell \text{Diff}_{can}(T^* Q) \]
\[ f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \quad (1) \]
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\[ \eta^* t \ell \dot{\ell} = -\£ X H \ell \quad (10) \]
\[ T^* Q X H \quad (11) \]
\[ g g^* \dot{\ell} = a d^* \delta H \delta \ell \Diff_{\text{can}}(T^* Q) \quad (12) \]
configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)$$

phase space $T^*Q$

$$-\dot{W}^\epsilon = \frac{1}{\epsilon} \{ p^\epsilon, W^\epsilon \} + \frac{1}{i} \{ p^\epsilon, W^\epsilon \} + O(\epsilon)$$

$$\dot{W}_a^0 + \{ \tau_a, W_a^0 \} = [W_a^0, F_a]$$

light transport equation

$$\dot{\ell} = -\{ \ell, H \}$$
Conservation of electromagnetic theory

Configuration space \( Q \subset \mathbb{R}^3 \)

\[
\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\mu} \nabla \times \frac{1}{\epsilon} \nabla \times \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)
\]

Phase space \( T^* Q \)

\[
-\dot{W}^\epsilon = \frac{1}{\epsilon} [p^\epsilon, W^\epsilon] + \frac{1}{\epsilon^2} \{p^\epsilon, W^\epsilon\} + O(\epsilon)
\]

\[\varepsilon \to 0\]

\[
\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]
\]

Fermat's principle

\[\hat{L} = n^2(q)\]
Fermat’s principle

\[ L = n(q) \]
geodesic energy \[ \hat{L} = n^2(q) \]

Fermat’s principle \[ L = n(q) \]
L(q, ˙q) = ω \sqrt{g_{i j} \dot{q}_i \dot{q}_j}

Fermat’s principle

\hat{L} = n^2(q)

Legendre transform

\hat{H} = g^{ij} \hat{p}_i \hat{p}_j

geodesic Hamiltonian

\hat{L} \xrightarrow{\text{Legendre transform}} \hat{H} = g^{ij} \hat{p}_i \hat{p}_j

geodesic energy

\hat{L} = n^2(q)

Fermat’s principle

\hat{L} = n(q)

L(n(q))

\hat{H} = c n(q) \parallel \hat{p} \parallel

light transport

H(\omega) = \hat{H} = c n(q) \parallel p \parallel

Legendre transform

\hat{H} = g^{ij} \hat{p}_i \hat{p}_j

geodesic Hamiltonian
\[ L = n^2(q) \quad \Rightarrow \quad \hat{L} = n^2(q) \]

Legendre transform

\[ \hat{H} = g^{ij} \hat{p}_i \hat{p}_j \]

\[ \sqrt{\hat{H}} = \frac{c}{n(q)} \| \hat{p} \| \]

Fermat’s principle

geodesic energy

geodesic Hamiltonian
Fermat's principle

\[ L = n(q) \]

geodesic energy

\[ \hat{L} = n^2(q) \]

Legendre transform

\[ \hat{H} = g^{ij} \hat{p}_i \hat{p}_j \]

geodesic Hamiltonian

\[ \sqrt{\hat{H}} = \frac{c}{n(q)} \| \hat{p} \| \]

light transport

\[ H = \omega \sqrt{\hat{H}} = \frac{c}{n(q)} \| p \| \]
electromagnetic theory

configuration space $Q \subset \mathbb{R}^3$

\[
\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & \frac{1}{\mu} \nabla \times \frac{1}{\epsilon} \nabla \times \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)
\]

phase space $T^*Q$

\[
-\dot{W}^\epsilon = \frac{1}{\varepsilon} \{ p^\epsilon, W^\epsilon \} + \frac{1}{i} \{ p^\epsilon, W^\epsilon \} + O(\varepsilon)
\]

\[
\varepsilon \rightarrow 0
\]

\[
\dot{W}^0_a + \{ \tau^0_a, W^0_a \} = [W^0_a, F_a]
\]

Fermat’s principle

\[
\hat{L} = n^2(q)
\]

Legendre transform

\[
\hat{\ell} = -\{ \ell, H \}
\]

light transport equation

microlocal analysis
(Wigner transform)

unpolarized radiation
configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t} \left( \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right)$$

| microlocal analysis (Wigner transform) |

phase space $T^*Q$

$$-\dot{W}^\epsilon = \frac{1}{\epsilon} \left[ p^\epsilon, W^\epsilon \right] + \frac{1}{i} \left\{ p^\epsilon, W^\epsilon \right\} + O(\epsilon)$$

$\varepsilon \to 0$

$$\dot{W}_a^0 + \left\{ \tau_a, W_a^0 \right\} = [W_a^0, F_a]$$

unpolarized radiation

Fermat’s principle

$$\dot{L} = n^2(q)$$

Legendre transform

light transport equation

$$\dot{\ell} = -\left\{ \ell, H \right\}$$

conservation of frequency
\[ \dot{\omega} + \mathcal{L}_v \omega = 0 \]
\[ \dot{\ell} + \mathcal{L}_X H \ell = 0 \]
\[ \dot{\ell} = a d^* \delta H \delta \ell = -\{\ell, H\} \] (13)

\[ \sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_{\mathcal{X}} f(x) \, dx \quad (14) \]

\[ S^* q Q \nu \bar{p} \] (15)

\[ P \eta t (P) \] (16)

\[ T^* q Q p \] (17)

\[ \text{Diff} \mu (Q) \text{Diff can} (T^* Q) \] (18)

\[ \dot{\omega} + \mathcal{L}_v \omega = 0 \quad \dot{\ell} + \mathcal{L}_X H \ell = 0 \] (19)

\[ g = X_{\text{Ham}} g^* \sim \text{Den}(T^* Q) \] (21)
\[ \sum_{n=1}^{\infty} \tilde{\omega} n + \dot{i} \ell f p = g_n \]
configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \nabla \times \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)$$

phase space $T^*Q$

$$-\dot{W}^\epsilon = \frac{1}{\epsilon} [p^\epsilon, W^\epsilon] + \frac{1}{i} \{p^\epsilon, W^\epsilon \} + O(\epsilon)$$

$$\dot{W}_a^0 + \{\tau_a, W_a^0 \} = [W_a^0, F_a]$$

Fermat's principle

$$\hat{L} = n^2(q)$$

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

photon mapping

reproducing kernel Galerkin projection

path tracing

finitary point functionals

classic iso-velocity description

S*Q = (T*Q\{0\})/\mathbb{R}^+
configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

phase space $T^*Q$

$$-\dot{W}^\varepsilon = \frac{1}{\varepsilon} [p^\varepsilon, W^\varepsilon] + \frac{1}{i} \{p^\varepsilon, W^\varepsilon\} + O(\varepsilon)$$

$$\varepsilon \to 0$$

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

Fermat’s principle

$$\hat{L} = n^2(q)$$

Legendre transform

right-cospere bundle reduction

transport theorem

$$S^*Q = (T^*Q \setminus \{0\}) / \mathbb{R}^+$$

classical iso-velocity description

measurements
The equation for the light transport equation is given by:

$$\frac{\partial}{\partial t} \left( \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{c} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right)$$

The conservation of frequency is given by:

$$\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]$$

Fermat's principle is:

$$\hat{L} = n^2(q)$$

The cosphere bundle reduction is:

$$S^* Q = \left( T^* Q \setminus \{0\} \right) / \mathbb{R}^+$$

Measurements are taken along the rays.

The Lie-Poisson structure of ideal light transport is:

$$\{p^e, W^e\} + O(\varepsilon)$$

The unpolarized radiation is:

$$\varepsilon \to 0$$

Finitary point functionals include:

- path tracing
- photon mapping
- radiosity
- classic iso-velocity description
- Fermat's principle

The electromagnetic theory includes:

- light transport equation
- cosphere bundle reduction
- transport theorem
\[ \frac{\partial}{\partial t} \left( \begin{array}{c} E \\ H \end{array} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{c} \nabla \times \\ \frac{1}{c} \nabla \times & 0 \end{array} \right) \left( \begin{array}{c} E \\ H \end{array} \right) \]

\[ -\dot{W}^\varepsilon = \frac{1}{\varepsilon} [p^\varepsilon, W^\varepsilon] + \frac{1}{i} \{p^\varepsilon, W^\varepsilon\} + O(\varepsilon) \]

\[ \dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a] \]

\[ \hat{L} = n^2(q) \]

light transport equation

\[ \dot{\ell} = -\{\ell, H\} \]

Fermat’s principle

cession of frequency

\[ S^Q = (T^*Q \setminus \{0\}) / \mathbb{R}^+ \]

cosphere bundle reduction

measurements

classical radiometry

\[ L(x, \omega) \cos \theta \, d\omega \, dA \]
electromagnetic theory

configuration space $Q \subset \mathbb{R}^3$

\[
\frac{\partial}{\partial t} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}
\]

\[\text{microlocal analysis (Wigner transform)}\]

phase space $T^*Q$

\[-\dot{W}^\epsilon = \frac{1}{\epsilon} [p^\epsilon, W^\epsilon] + \frac{1}{i} \{p^\epsilon, W^\epsilon\} + O(\epsilon)
\]

\[\varepsilon \to 0\]

\[\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]\]

\[\text{unpolarized radiation}\]

Fermat’s principle

\[\hat{L} = n^2(q)\]

\[\text{Legendre transform}\]

light transport equation

\[\dot{\ell} = -\{\ell, H\}\]

\[\text{cosphere bundle reduction}\]

classic iso-velocity description

\[S^*Q = (T^*Q \setminus \{0\}) / \mathbb{R}^+\]

\[\text{measurements}\]

\[L(x, \omega) \cos \theta d\omega dA\]

\[\text{conservation of } \ell \text{ along “rays”}\]
configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)$$

phase space $T^*Q$

$$-\dot{W}^\varepsilon = \frac{1}{\varepsilon} [p^\varepsilon, W^\varepsilon] + \frac{1}{i} \{p^\varepsilon, W^\varepsilon\} + O(\varepsilon)$$

$$\dot{W}_a^0 + \{\tau_a, W_0^a\} = [W_0^a, F_a]$$

Fermat's principle

$$\hat{L} = n^2(q)$$

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

classical iso-velocity description

$$S^*Q = (T^*Q \setminus \{0\})/\mathbb{R}^+$$

measurements

classical radiometry

$$L(x, \omega) \cos \theta d\omega dA$$

conservation of frequency

unpolarized radiation

conservation of frequency along "rays"
\[ \dot{\ell} = -\mathcal{L}_{X_H} \ell \]

\[ \eta^*_H \ell \]

\[ X_H \]

\[ T^*Q \]

\[ \sum_{i=1}^n \sum_{t} \frac{\dot{x}}{\dot{x}} \]

\[ \dot{\lambda} = \sum_{i=1}^n \phi_i \]

\[ \sum_{i=1}^n \sum_{t} \frac{\dot{x}}{\dot{x}} \]

\[ \delta^\ell \]

\[ (\delta^\ell)_n \]

\[ \langle \mathcal{Q} \rangle_{\ell} \]

\[ \sum_{i=1}^n \sum_{t} \frac{\dot{x}}{\dot{x}} \]

\[ \delta^\ell \]

\[ (\delta^\ell)_n \]

\[ \langle \mathcal{Q} \rangle_{\ell} \]
\[ H(q,p) = c_n(q) \parallel p \parallel (22) = \parallel p \parallel c_n(q) p \parallel p \parallel (23) \]
\[ \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} = c_n(q) p \parallel p \parallel (24) \]
\[ -\frac{\partial p}{\partial t} = \frac{\partial H}{\partial q} = c_\parallel p \parallel \nabla q n(q) n^2(q) (25) \]
\[ \dot{\ell} = -\{ \ell, c_n(q) \parallel p \parallel \} (26) \]
\[ \dot{W}_\epsilon = -\{ p_\epsilon, W_\epsilon \} (27) \]
\[ Q \subset \mathbb{R}^3 (28) \]
\[ X_{\text{div}} (29) \]
\[ X_{\text{Ham}} (30) \]
\[ \dot{\omega} = \text{ad}^{\ast} \delta H \delta \omega (32) \]
\[ \dot{\omega} \]
$H(q,p) = cn(q) \| p \| (22)$

$\partial q / \partial t = \partial H / \partial p = cn(q) \| p \| (24)$

$- \partial p / \partial t = \partial H / \partial q = c \| p \| \nabla q n(q) (25)$

$\dot{\ell} = - \{ \ell, cn(q) \| p \| \} (26)$

$\dot{W}_\epsilon = - \{ p_\epsilon, W_\epsilon \} (27)$

$Q \subset \mathbb{R}^3 (28)$

$X = \text{div} \text{Ham} (30)$

$Q_g = X \text{div} \phi_t, \ t \in \text{Diff}_\mu(Q) (31)$

$\dot{\omega} = a d^* \delta H \delta \omega (32)$

$\psi_3$
\( H(q, p) = c_n(q) \| p \| \) \( (22) \)

\( \partial q / \partial t = \partial H / \partial p = c_n(q) \| p \| \) \( (24) \)

\( -\partial p / \partial t = \partial H / \partial q = c_\| p \\| \nabla q n(q) \) \( n^2(q) \) \( (25) \)

\( \dot{\ell} = -\{ \ell, c_n(q) \| p \| \} \) \( (26) \)

\( \dot{W}_\varepsilon = -\{ \{ p_\varepsilon, W_\varepsilon \} \} \) \( (27) \)

\( Q \subset \mathbb{R}^3 \) \( (28) \)

\( \text{Xdiv (29)} \)

\( \text{XHam (30)} \)

\( Q_g = \text{Xdiv} \phi_t \in \text{Diff} \mu(Q) \) \( (31) \)

\( \dot{\omega} = ad^{\ast} \delta H \delta \omega \omega \) \( (32) \)
\[ H(q,p) = c_n(q) \parallel p \parallel (22) = \parallel p \parallel c_n(q) \parallel p \parallel (23) \]

\[ \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} = c_n(q) \parallel p \parallel (24) \]

\[ -\frac{\partial p}{\partial t} = \frac{\partial H}{\partial q} = c \parallel p \parallel \nabla q_n(q)^2 (q) (25) \]

\[ \dot{\ell} = -\left\{ \ell, c \parallel q \parallel \right\} (26) \]

\[ \dot{W}_{\varepsilon} = -\left\{ p_{\varepsilon}, W_{\varepsilon} \right\} (27) \]

\[ Q \subset \mathbb{R}^3 (28) \]

\[ X_{\text{div}} (29) X_{\text{Ham}} (30) \]

\[ Q_g = X_{\text{div}} \phi_t \in \text{Diff} \mu(Q) (31) \]

\[ \dot{\omega} = a d^* \delta H \delta \omega \omega (32) \]
\[ H(q, p) = c_n(q) \| p \| (22) = \| p \| c_n(q) \| p \| (23) \]

\[ \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} = c_n(q) \| p \| (24) \]

\[ -\frac{\partial p}{\partial t} = \frac{\partial H}{\partial q} = c_\| p \| \nabla q (25) \]

\[ \dot{\ell} = -\{\ell, c_n(q) \| p \|\} (26) \]

\[ \dot{W}_\epsilon = -\{\{p_\epsilon, W_\epsilon\}\} (27) \]

\[ Q \subset \mathbb{R}^3 (28) \]

\[ \mathcal{X} \text{div} (29) \]

\[ \mathcal{X} \text{Ham} (30) \]

\[ Q_g = \mathcal{X} \text{div} \phi_t \in \text{Diff}_\mu(Q) (31) \]

\[ \dot{\omega} = a d^* \delta H \delta \omega (32) \]
\[ \dot{\omega} = -\mathcal{L} v \omega \] (42)

\[ \dot{\ell} = -\{\ell, H\} \] (43)

\[ \text{Diff}_{\text{can}}(T^*Q) \]

\[ L(x, \omega) \nabla_x L(x, \omega) = 0 \]

\[ L(x, \bar{\omega}) = \int H(x) L(x, \omega) \rho_x(\omega, \bar{\omega}) d\omega \] (45)

\[ Q = G = \text{Diff}_{\text{can}}(T^*Q) \] (46)
\[ \dot{\omega} = -\mathbf{v} \omega \quad (42) \]
\[ \dot{\ell} = -\{\ell, H\} \quad (43) \]
\[ \text{Diff}_{\text{can}}(T^*Q) \]
\[ L(x, \omega) \nabla_x L(x, \omega) = 0 \]
\[ L(x, \bar{\omega}) = \int H \rho_x(\omega, \bar{\omega}) d\omega \quad (45) \]
\[ Q = G = \text{Diff}_{\text{can}}(T^*Q) \quad (46) \]
\[ (T_g R_{g^{-1}}) V_t = v_t \]

\[ TG \quad (T_g R_{g^{-1}}) V_t = v_t \quad g_+ \]

Lagrangian \quad right \quad translation \quad Eulerian

\[ T^*G \quad g^*_+ \]
\[ \mathfrak{g}_- \quad \mathcal{V}_t = (T_g L_{g^{-1}}) V_t \quad T^*G \quad (T_g R_{g^{-1}}) V_t = v_t \quad \mathfrak{g}_+ \]

Convective

left translation

Lagrangian

right translation

Eulerian

\[ \mathfrak{g}^*_\quad \quad \quad T^*G \quad \quad \quad \mathfrak{g}^*_+ \]
\[ \mathcal{V}_t = (T_g L_{g^{-1}}) V_t \quad T G \quad (T_g R_{g^{-1}}) V_t = v_t \]

\[ \mathfrak{g}_- \quad \mathfrak{g}^*_+ \quad \mathfrak{g}^*_+ \]

Convective \quad \text{left translation} \quad \text{Lagrangian} \quad \text{right translation} \quad \text{Eulerian}

right invariant Hamiltonian
\( g_- \quad \mathcal{V}_t = (T_g L_{g^{-1}}) V_t \quad g_+ \)

Convective  \( \text{left translation} \)  Lagrangian  \( \text{right translation} \)  Eulerian

\( g^*_- \quad T^*G \quad g^*_+ \)

right invariant Hamiltonian

\( J_L = T^*_e R \)
emergence of light transport
configuration space \( \mapsto \mathbb{R}^3 \)
\[
\frac{\partial}{\partial t} \left( \frac{\mathbf{E}}{\mathbf{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \frac{1}{\mu} \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \frac{\mathbf{E}}{\mathbf{H}} \right)
\]
\[\text{microlocal analysis (Wigner transform)}\]

phase space \( T^*Q \)
\[
-\dot{W}_\epsilon = \frac{1}{\epsilon} \{p^\epsilon, W^\epsilon\} + \frac{1}{i} \{p^\epsilon, W^\epsilon\} + O(\epsilon)
\]
\[\varepsilon \to 0\]

\[\dot{W}_a + \{\tau_a, W^0_a\} = [W^0_a, F_a]\]

Fermat’s principle
\[\hat{L} = n^2(q)\]

light transport equation
\[\dot{\ell} = -\{\ell, H\}\]

Lie-Poisson structure of ideal light transport
\[\dot{\ell} = X_{\mathcal{H}_\ell}[\ell] = -\text{ad}_{\mathcal{H}_\ell}^{\epsilon}(\ell) = -\{\ell, H\}\]

classical iso-velocity description
\[S^*Q = (T^*Q \setminus \{0\}) / \mathbb{R}^+\]

measurements
\[L(x, \omega) \cos \theta \, d\omega \, dA\]

conservation of frequency
unpolarized radiation

conservation of frequency

conservation of frequency
\[ \mathcal{V}_t = (T_g L_{g^{-1}}) V_t \]

Convective translation

\[ T^*G \]

Lagrangian translation

Eulerian

\[ \mathfrak{g}_- \rightarrow \mathcal{V}_t = (T_g L_{g^{-1}}) V_t \rightarrow TG \rightarrow (T_g R_{g^{-1}}) V_t = v_t \rightarrow \mathfrak{g}_+ \]

\[ \mathfrak{g}^*_- \rightarrow T^*G \rightarrow \mathfrak{g}^*_+ \]

Convective right invariant

\[ J_L = T^*_e R \]

Hamiltonian dynamics
\[ \nabla_t = \left( T_g L_{g^{-1}} \right) V_t \quad \text{right translation} \]

\[ \left( T_g R_{g^{-1}} \right) V_t = v_t \quad \text{Eulerian} \]

**Convective**

\[ g_- \quad T^*G \quad g_+ \]

**Lagrangian**

\[ g^* \quad T^*G \quad g^*_+ \]

**Eulerian**

\[ \mathbf{J}_R = T_e^* L \quad J_L = T_e^* R \]

**Noether's theorem**

**right invariant Hamiltonian**

**Dynamics**
\[ \ell = a \delta H \frac{\delta \ell}{\ell} = -\{\ell, H\} \quad (13) \]

\[ \sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_{X} f(x) dx \quad (14) \]

\[ S^* q \quad (15) \]

\[ P^t(P) \quad (16) \]

\[ T^* q \quad (17) \]

\[ \text{Diff} \mu(Q) \quad \text{Diff} \quad (18) \]

\[ \dot{\omega} + \mathcal{L}_v \omega = 0 \quad \dot{\ell} + \mathcal{L}_X H \ell = 0 \quad (19) \]

\[ g = X_{\text{Ham}} g^* \quad (21) \]
\[ \dot{\ell} = a d^* \delta H \delta \ell \]

\[ \ell = \{ \ell, H \} \quad (13) \]

\[ \sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_X f(x) dx \quad (14) \]

\[ S^* Q \bar{\nu} \bar{p} (15) \]

\[ P_{\eta t}(16) \]

\[ T^* Q p (17) \]

\[ \text{Diff} \mu (Q) \text{Diff} \text{can} (T^* Q) (18) \]

\[ \dot{\omega} + \varepsilon \omega = 0 \quad \dot{\ell} + \varepsilon X H \ell = 0 \quad (19) \]

\[ g = X \text{Ham} g^* \sim \text{Den}(T^* Q) \quad (21) \]
\[ \dot{\ell} = a \delta H \delta \ell \]
\[ \dot{\ell} = -\{\ell, H\} \]
\[ n \sum_{i=1}^n f(\lambda_i) p(\lambda_i) \rightarrow \int_X f(x) \, dx \]
\[ S^* \bar{Q} \bar{p} \]
\[ P_{\eta t}(P) \]
\[ T^* Q p \]
\[ \text{Diff} \mu (Q) \text{Diff} \]
\[ \dot{\omega} + \varepsilon X H \ell = 0 \]
\[ g = X \text{Ham} g^* \sim = \text{Den}(T^* Q) \]
\[ \ell = a \ast \delta H \delta \ell \]
\[ \ell = \{\ell, H\} \] (13)
\[ \sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_{X} f(x) dx \] (14)
\[ S^* \overline{Q} \nu \overline{p} \] (15)
\[ P_{\eta t} (P) \] (16)
\[ T^* Q \] (17)
\[ \text{Diff} \mu (Q) \text{Diff} \text{can} (T^* Q) \] (18)
\[ \dot{\omega} + \mathcal{L}_v \omega = 0 \]
\[ \dot{\ell} + \mathcal{L}_X H \ell = 0 \] (19)
\[ X_{\text{Ham}} g^* \sim = \text{Den}(T^* Q) \] (21)
\[\dot{\ell} = a d^* \delta H \delta \ell = -\{\ell, H\} \quad (13)\]

\[\sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_{X} f(x) \, dx \quad (14)\]

\[S^* q Q \bar{\nu} \bar{p} \quad (15)\]

\[P \eta (P) \quad (16)\]

\[\dot{q} = \dot{H} \quad (17)\]

\[\text{Diff} \mu (Q) \text{Diff} \quad (18)\]

\[\dot{\omega} + \ell v \omega = 0 \quad \dot{\ell} + \ell X H \ell = 0 \quad (19)\]

\[g = \text{Ham} g^* \sim = \text{Den}(T^* Q) \quad (21)\]
\[ \ell = a \delta \quad (13) \]

\[ \sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_X f(x) \, dx \quad (14) \]

\[ S^* \bar{Q} \bar{p} \quad (15) \]

\[ \mathbb{P} \eta \quad (16) \]

\[ T^* Q \quad (17) \]

\[ \text{Diff} \quad (18) \]

\[ \dot{\omega} + \xi \omega = 0 \quad (19) \]

\[ X \quad (20) \]

\[ g = X \text{Ham} g^* \sim \text{Den}(T^* Q) \quad (21) \]
\[
\dot{\omega} + \varepsilon v \omega = 0
\]
\[
\dot{\ell} + \varepsilon X H \ell = 0
\]
\[ \dot{\ell} = a \delta H \delta \ell \]

\[ \ell = -\{\ell, H\} \]

\[ \sum_{i=1}^{n} f(\lambda_i) p(\lambda_i) \rightarrow \int_X f(x) dx \]

\[ S^* q \bar{Q} \nu \bar{p} \]

\[ P \eta_t \]

\[ T^* q \bar{Q}p \]

\[ \text{Diff} \]

\[ \dot{\omega} + \£_v \omega = 0 \]

\[ \dot{\ell} + \£_X H \ell = 0 \]

\[ g = \text{Ham} g \sim = \text{Den} \left( T^* Q \right) \text{Diff}_{\text{can}} \left( T^* Q \right) \]
configuration space \( Q \subset \mathbb{R}^3 \)

\[
\frac{\partial}{\partial t} \left( \vec{E} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \left( \frac{1}{\mu} \nabla \times \vec{E} \right) \end{array} \right)
\]

\( \frac{\partial}{\partial t} \left( \vec{H} \right) \)

\[
\frac{\partial}{\partial t} \left( \vec{H} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \left( \frac{1}{\mu} \nabla \times \vec{E} \right) \end{array} \right)
\]

\( \frac{\partial}{\partial t} \left( \vec{H} \right) \)

phase space \( T^*Q \)

\[
-\dot{W}^\epsilon = \frac{1}{\epsilon} \left[ p^\epsilon, W^\epsilon \right] + \frac{1}{i} \left\{ p^\epsilon, W^\epsilon \right\} + O(\epsilon)
\]

\[
\dot{W}_a^0 + \left\{ \tau_a, W_a^0 \right\} = \left[ W_a^0, F_a \right]
\]

\( \dot{\ell} = -\left\{ \ell, H \right\} \)

Fermat's principle

\( \hat{L} = n^2(q) \)

Legendre transform

light transport equation

\( \dot{\ell} = -\left\{ \ell, H \right\} \)

Lie-Poisson structure of ideal light transport

\( \dot{\ell} = X_{\mathcal{H}_\ell} [\ell] = -\text{ad}_{\mathcal{H}_\ell}^{*} (\ell) = -\left\{ \ell, H \right\} \)

conservation of frequency

unpolarized radiation

conservation of frequency

measurements

classical iso-velocity description

\( S^*Q = (T^*Q \setminus \{0\}) / \mathbb{R}^+ \)

cosphere bundle reduction

transport theorem

ideal light transport: globally defined Hamiltonian vector field

\( L(x, \omega) \cos \theta d\omega dA \)

Lie-Poisson structure of ideal light transport

\( \dot{\ell} = X_{\mathcal{H}_\ell} [\ell] = -\text{ad}_{\mathcal{H}_\ell}^{*} (\ell) = -\left\{ \ell, H \right\} \)

conservation of \( \ell \) along “rays”
<table>
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<th>ideal light transport</th>
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<td>Kelvin’s circulation theorem</td>
<td>conservation of radiance</td>
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configuration space \( Q \subset \mathbb{R}^3 \)

\[
\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\epsilon} \nabla \times & 0 \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)
\]

phase space \( T^*Q \)

\[
-\dot{W}^\epsilon = \frac{1}{\epsilon} [p^\epsilon, W^\epsilon] + \frac{1}{i} \{p^\epsilon, W^\epsilon\} + O(\epsilon)
\]

\( \epsilon \to 0 \)

\[
\dot{W}_a^0 + \{\tau_a, W^0_a\} = [W^0_a, F_a]
\]

Fermat’s principle

\( \hat{L} = n^2(q) \)

Legendre transform

light transport equation

\( \dot{\ell} = -\{\ell, H\} \)

classical iso-velocity description

\( S^*Q = (T^*Q\setminus\{0\})/\mathbb{R}^+ \)

measurements

classical radiometry

\( L(x, \omega) \cos \theta \, d\omega \, dA \)

Lie-Poisson structure of ideal light transport

\( \dot{\ell} = \mathcal{X}_{\mathcal{H}_\ell} [\ell] = -\text{ad}_{\mathcal{H}_\ell}^* (\ell) = -\{\ell, H\} \)

conservation of \( \ell \) along “rays”
 Electromagnetic theory

\[
\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\mu} \nabla \times \varepsilon \nabla \times \mu \nabla \times \nabla \times \varepsilon \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)
\]

configuration space \( Q \subset \mathbb{R}^3 \)

\[
-\dot{\bar{W}}^\varepsilon = \frac{1}{\varepsilon} [p^\varepsilon, W^\varepsilon] + \frac{1}{i} \{ p^\varepsilon, W^\varepsilon \} + O(\varepsilon)
\]

\( \dot{\bar{W}}_a^0 + \{ \tau_a, W_a^0 \} = [W_a^0, F_a] \)

phase space \( T^*Q \)

Fermat's principle \( \hat{L} = n^2(q) \)

Lie-Poisson structure of ideal light transport

\[
\dot{\ell} = X_{\mathcal{H}_\ell}[\ell] = -\text{ad}_{\mathcal{H}_\ell}^* (\ell) = -\{ \ell, H \}
\]

\( \ell_t = U_t \ell_0 \)

classical iso-velocity description

\( S^*Q = (T^*Q \setminus \{0\}) / \mathbb{R}^+ \)

measurements

\( L(x, \omega) \cos \theta d\omega dA \)
configuration space \( Q \subset \mathbb{R}^3 \)

\[
\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\mu} \nabla \times \\ \frac{1}{

\text{phase space } T^*Q \n
\dot{W}^e = \frac{1}{\varepsilon} [p^e, W^e] + \frac{1}{i} \{p^e, W^e\} + O(\varepsilon) \n
\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a] \n
\text{Fermat's principle } \n \hat{L} = n^2(q) \n
\text{Lie-Poisson structure of ideal light transport } \n \dot{\ell} = X_{\mathcal{H}_\ell}[\ell] = -\text{ad}_{\mathcal{H}_\ell}^{\frac{\mu}{\varepsilon}}(\ell) = -\{\ell, H\} \n
\ell_t = \mathcal{U}_t \ell_0 \n
\mathcal{T} = \bar{\mathcal{U}} \mathcal{R}_\rho \n
\text{operator formulation of light transport } \n \bar{\ell} = \ell_0 + \mathcal{T}^1 \ell_0 + \mathcal{T}^2 \ell_0 + \ldots
Thanks to

Alex Castro, Eugene Fiume, Tyler de Witt, Mathieu Desbrun, Tudor Ratiu, Boris Khesin, and Jerry Marsden.

lessig@caltech.edu
http://users.cms.caltech.edu/~lessig/dissertation/
http://arxiv.org/abs/1206.3301
Electromagnetic theory

Configuration space $Q$

$$\frac{\partial}{\partial t} \left( \frac{\vec{E}}{\vec{H}} \right) = \left( \begin{array}{cc} 0 & -\frac{1}{\epsilon} \nabla \times \\ \frac{1}{\mu} \nabla \times & 0 \end{array} \right) \left( \frac{\vec{E}}{\vec{H}} \right)$$

Phase space $T^*Q$

$$-\dot{W}^\epsilon = \frac{1}{\epsilon} \{ p^\epsilon, W^\epsilon \} + \frac{1}{i} \{ p^\epsilon, W^\epsilon \} + O(\epsilon)$$

$$\varepsilon \to 0$$

Unpolarized radiation

Conservation of frequency

Fermat's principle

$$\hat{L} = n^2(q)$$

Legendre transform

Light transport equation

$$\dot{\ell} = -\{ \ell, H \}$$

Lie-Poisson structure of ideal light transport

$$\dot{\ell} = X_{\mathcal{H}_t}[\ell] = -ad^*_{\nabla\mathcal{H}_t}(\ell) = -\{ \ell, H \}$$

Stone's theorem

$$\ell_t = U_t \ell_0$$

Classical iso-velocity description

$$S^*Q = (T^*Q \setminus \{ 0 \}) / \mathbb{R}^+$$

Measurements

Classical radiometry

$$L(x, \omega) \cos \theta \, d\omega \, dA$$

Inclusion of surface scattering

Operator formulation of light transport

$$\tilde{\ell} = \ell_0 + T^1 \ell_0 + T^2 \ell_0 + \ldots$$