A gentle introduction to Microswimming:

geometry, physics, analysis

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Why study microswimming?

Feynman:  there is plenty of room in the bottom!

(and plenty of grant money, so it seems)
In the last 15 years:

New tools for particle visualization, in vivo cell manipulation, biochemical structure, genomics and function.

These developments are bringing new challenges and opportunities for the applied mathematician to do collaborative work with biologists and engineers.

One example:

Recent experiments in R. Goldstein, DAMPT
Collaborators

Kurt Ehlers and Richard Montgomery

Joaquim Delgado

Marco Raupp, Alexandre Cherman, Gerusa Araujo, Fernando Duda

Advice/ suggestions

Howard Berg, Theodore Wu, Moyses Nussenzweig

Greg Huber, Scott Kelly, John Bush, Lisa Fauci, Peko Hosoi, ...

Encouragement:
Collaborators

Kurt Ehlers and Richard Montgomery

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Advice/ suggestions

Howard Berg, Theodore Wu

Greg Huber, Scott Kelly,

Encouragement: Jerry Marsden [ JK, KE, RM Problems and Progress ]
Microswimming is governed by Stokes equations on an incompressible fluid.

**Taylor (movie)**

Ambient: $R^2$ (life at interface) or $R^3$

“Molasses Laplacian”

$$0 = - \nabla p + \mu \Delta u$$

$$\text{div} \ u = 0$$

**Purcell Life at low $Re$**

Reynolds = $O(10^{-5})$
Stress tensor \( T = -pl + \sigma \), \( \sigma = ?? \)

\[
\nabla u = \begin{bmatrix}
\partial_z u & \partial_y u & \partial_x u \\
\partial_z v & \partial_y v & \partial_x v \\
\partial_z w & \partial_y w & \partial_x w
\end{bmatrix}
\]

denote the Jacobian matrix of \( u \). By Taylor’s theorem,

\[
u(y) = u(x) + \nabla u(x) \cdot h + O(h^2),
\]

where \( \nabla u(x) \cdot h \) is a matrix multiplication, with \( h \) regarded as a column vector. Let

\[
D = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right].
\]

This is reasonable, because when a fluid undergoes a rigid body rotation, there should be no diffusion of momentum.

3. \( \sigma \) is symmetric. This property can be deduced as a consequence of balance of angular momentum.

Since \( \sigma \) is symmetric, if follows from properties 1 and 2 that \( \sigma \) can depend only on the symmetric part of \( \nabla u \); that is, on the deformation \( D \). Because \( \sigma \) is a linear function of \( D \), \( \sigma \) and \( D \) commute and so can be simultaneously diagonalized. Thus, the eigenvalues of \( \sigma \) are linear functions of those of \( D \). By property 2, they must also be symmetric because we can choose \( U \) to permute two eigenvalues of \( D \) (by rotating through an angle \( \pi/2 \) about an eigenvector), and this must permute the corresponding eigenvalues of \( \sigma \). The only linear functions that are symmetric in this sense are of the form

\[
\sigma_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i, \quad i = 1, 2, 3,
\]

where \( \sigma_i \) are the eigenvalues of \( \sigma \), and \( d_i \) are those of \( D \). This defines the constants \( \lambda \) and \( \mu \). Recalling that \( d_1 + d_2 + d_3 = \text{div} u \), we can use property 2 to transform \( \sigma_i \) back to the usual basis and deduce that

\[
\sigma = \lambda(\text{div} u)I + 2\mu D,
\]

where \( I \) is the identity. We can rewrite this by putting all the trace in one term:

\[
\sigma = 2\mu[D - \frac{1}{3}(\text{div} u)I] + \zeta(\text{div} u)I
\]

where \( \mu \) is the first coefficient of viscosity, and \( \zeta = \lambda + \frac{\mu}{3} \) is the second coefficient of viscosity.

source: Marsden/Chorin
An organism/robot is a deforming boundary immersed in the ambient.

There are physical requirements for self propulsion.

What are them?  

( nội a couple slides.)

For now: \[ T = -\rho \mathbf{l} + 2 \mu \mathbf{D} \]

\[ \text{div } T = 0 \quad \text{and} \quad \text{div } \mathbf{u} = 0 \]

\[ \mathbf{u} \to 0 \quad \text{at infinity} \quad , \]

no slip condition imposed on all boundaries  

(moving or fixed)
Ambient: $\mathbb{R}^2$ (life at interface) or $\mathbb{R}^3$

but ... everything has boundaries !!!!

Common wisdom:

**Boundaries affect motion substantially only**

when organisms are close to them.
Geometry and Physics of Microswimming *

(Purcell & Shapere-Wilczek)

It's a Gauge theory!

Key words: shape space, principal bundle, connection!

And a subriemannian geometry!

Metric is the hydrodynamical power efficiency notions

(For collective swimming: Ehresmann connection)

Recife notes
www.impa.br/~jair

* Taylor and Lighthill already knew in the 1950’s what it was all about. Later on, analysts occasionally make blunders (see O.P. (2.14))
Microswimming is a gauge theory!!

Purcell’s 3 linked swimmer
(only recently studied)

toroidal animal (Taylor, Purcell)

Figure 1a. Hypothetical ring-shaped animal capable of rotating its body in the direction indicated. b. Direction of motion when the ring rolls on the outside of a cylinder. c. Direction of motion when the ring rolls on the inside of a cylindrical tube.
What is the metric? Hydrodynamical power expenditure *

$U = \text{vectorfield along the boundary}$

$u = \text{solution of exterior Stokes equations (analogous to Dirichlet problem for Laplacian)}$

$\sigma = \text{stress tensor associated to } u \quad F = \sigma \cdot n \text{ along the boundary}$

integrate $F \cdot U$ on $S$, call it $<<U,U>>$

$P : U \longrightarrow F \quad \text{Resistance operator}$

symmetric \hspace{1.5cm} \text{(Lorenz reciprocity)}$

* Discuss the envelope approximation
A “wet” Calderon problem?

\[ P : U \to F \] analogous to “Dirichlet to Neumann”

It is well known (see, e.g., [21]) that, given \( f \in C^{2,\alpha}(\Omega) \), there exists a unique solution of the boundary-value problem

\[
\begin{aligned}
\nabla \cdot (\gamma(x,u) \nabla u) &= 0 \quad \text{in } \Omega, \\
\left. u \right|_{\partial\Omega} &= f.
\end{aligned}
\] (19.164)

We define the Dirichlet to Neumann map \( \Lambda_{\gamma} : C^{2,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega) \) as the map given by

\[
\Lambda_{\gamma} : f \to \left. \nu \cdot \gamma(x,f) \nabla u \right|_{\partial\Omega},
\] (19.165)

where \( u \) is the solution of (19.164) and \( \nu \) denotes the unit outer normal of \( \partial\Omega \).

Physically, \( \gamma(x,u) \) represents the (anisotropic, quasilinear) conductivity of \( \Omega \) and \( \Lambda_{\gamma}(f) \) the current flux at the boundary induced by the voltage \( f \).

We study the inverse boundary-value problem associated to (19.164): how much information about the coefficient matrix \( \gamma \) can be obtained from knowledge of the Dirichlet to Neumann map \( \Lambda_{\gamma} \)?

Marsden on Calderon
What is the Connection?

Horizontal spaces:

physically allowed motions for self propulsion

Total force = 0  Total torque = 0

Vertical spaces: rigid motions (with frozen shape)

A key fact: Vertical ⊥ Horizontal

Proof.

“Lorenz reciprocity” (Green like identities for the “gooey“ Laplacian)

Answer: Its the mechanical connection
The connection 1 form solves the self rotating torus

(Taylor, Purcell)

Hold the shape in place,
SR boundary conditions: solve Stokes equations, compute total force $F$ and total torque $T$ (most likely $T = 0$).

RB counterflow: solve Stokes equations with unit velocity. Compute total force and adjust to have minus the force calculated previously.

You can invent other animals that have no trouble swimming. We had better be able to invent them, since we know they exist. One you might think of first as a physicist, is a torus. I don't know whether there is a toroidal animal, but whatever other physiological problems it might face, it clearly could swim at low Reynolds number

Purcell,
Life at low Reynolds
Figure 1a. Hypothetical ring-shaped animal capable of rotating its body in the direction indicated. b. Direction of motion when the ring rolls on the outside of a cylinder. c. Direction of motion when the ring rolls on the inside of a cylindrical tube.

The action of waving cylindrical tails in propelling microscopic organisms

By Sir Geoffrey Taylor, F.R.S.

(Received 3 October 1951)
**Spirochetes**


Self rotation induced flow (depicted in the figure)
Rigid Body counterflow with the opposite total torque
(note that total force vanishes)

Lighthill’s analysis

More later in this talk

*Schematic figure showing the mechanism for the self rotation about a local body axis.*

* Different trick by spiroplasma (Greg Huber)*
How to compute the curvature?

( Some tricks of the trade that we learned in the 1990’s )

Is it really needed to solve Stokes equations for any deformation?

Answer: yes and no.

Yes, we need to extend vectorfields defined on the boundary of the shape as the external Stokes flows; we need to Lie bracket them.

No, if we need only the connection 1 form.

Shapere/Wilzek recipie and more explained (see Recife lectures)
With curvature form, get Propulsion operator $F$ (antisymmetric)

$F(U,V) =$ infinitesimal displacement generated by $U,V, -U, -V$

An element of $sE(3)$
The “Curvature Approximation Formula” (Shapere/Wilczek) Concerns small deformations of an “average shape” (or envelope model)

- Geometric thinking organizes the Stokes flows calculations.

(Lie brackets “lurk” in the papers by the founding fathers)
Optimization

Min $P$, subject to prescribed $F$

For small deformations of amplitude $\varepsilon$

of an "average shape" $s_0$

We get a linear algebra problem (in infinite dimension)

Some efficiency concepts were given in Shapere/Wilczek (others recently)

See discussion in JK/J.Delgado on "Pareto optimization".
Purcellian Mechanics

1. **Cell & flagellum**  **Bacterial motor**  (Berg)

2. Two linked swimmer

   (groups of Peko Hosoi, Howard Stone, Greg Huber ...)

   One of the fundamental axiom: for rods $F_\perp = 2 F_{\parallel}$

3. Axiomatization: resistance matrices add; equivariance

Geometric Mechanics of **N-linked Swimmers**  (with Gerusa Araujo)
Problem 1  (hopefully not too hard).

Apply Pontryagin, what are the optimal patterns?

... and in the continuous limit?

do you get progressive waves

of arcs of circles from base to tip?
The holy grail: how molecular motors (dyneyn) act?

Problem 2 (hard): Incorporate internal forces in the modeling.

Use a Geometric Mechanics approach to organize the analysis.

Can you infer what are the internal forces from the movies?

Start with internal force fields with biological interpretation; how do they relate with the stress tensor at the solid fluid contact?

C. Brokaw

H. Gadelha

Comment: immersed boundary method

internal dissipation

How to start this program? Lighthill (J. Eng. Math., 1996)
Distributed molecular motors of the eukariote flagella

Charles Brokaw  Microtubule sliding

Bending patterns

Question: are optimal patterns waves formed by arcs of circles? (or near to)
Spirochetes revisited

Problem 3  (defy the experts!)

A 6 pack of beer offered !!

Encapsulated propulsion mechanisms

Myxo

Internal helical flagellum

Spirochete on a box

Would you like to swallow this?

Hirose lab movie
Swimming in “Fatland” \( (S^n, \ n = 2 \text{ or } 3) \)

Problem 4*: Topology matters?

* Another six-pack of beer for the first answer - does not need to be correct.
Swimming in “Fatland” \((S^n, \ n = 2\ or\ 3)\)

Topology matters? Does it prevent swimming?

[ I hope not. ] Rewrite the connection condition in terms of a Momentum map

\[ J: T^*Q \rightarrow g^* \]

\[ Q = \{ \text{embeddings of reference body } B \text{ in } S^n \}, \]

\[ G = SO(n) \text{ acting in targets } \sum = q(B) \text{ by rigid motion} \]

Identify \(TQ \equiv T^*Q\) by Power metric

Microswimming is just like the cart flip: \( J = 0 \).]
Final Remarks I. Analysis / numerics: tools for Stokes flows

biharmonic equation / Darboux representation (2d)

slender body approximations (Lighthill)

(multi)pole collocation methods (Wu, Weinbaum, ...)

regularized Stokeslets (Cortez)

immersed boundary method (Peskin)

boundary integral methods (Pozrikidis)

.......... 

(we will discuss none of them!)

ESCAPE ROUTE: Taylor waving sheet + tangent plane approximation
Final Remark II. Can you one-up the Scallop Paradox?

Acoustic streaming

Tuning fork in molasses

Play a guitar under water:

For MEMS devices

Quartz tuning fork
(cost: $ 10 )

just a cookie:

“Snapping shrimp” (Detlef Lohse)
Final Remark III (final) Taylor’s waving sheet and the tangent plane approximation

**TPA:** Analogous to the planar wave superposition for Laplacian operator.

**Taylor’s waving sheet**

**Kurt Ehlers 8th order**

**Application:** Synechococcus locomotion

**Acoustic streaming?** 2.5 more efficient
Taylor waving sheet revisited recently!!

**Wu** (not so recent, 1961)

**Kozlov-Ramodanov** (2002, potential flows + “recoil”)  
**Kozlov-Onischenko** (2004)

**Childress IMAtalk SC-Spag-Tokieda** (all Reynolds + “recoil”)

**Eric Lauga** (transient) **phaselocking** (cooperation) **higher order**

**Kurt calculations** Question: why only even terms are present?

**Annete Hosoi-Wilkening** **Annete-Chan**
The tangent plane approximation

Can be used on problems where a “local” wave can be identified.
Helical surface waves may explain the mystery of *Synechococcus* swimming

Our recent work

(JK, Kurt Ehlers)  (KE, G.Oster)

`spheroid.mov`
The mysterious open sea swimmer Synechococcus

(Waterbury et al., 1985)
Food for thought.

Research on autonomous micro swimming devices is attracting great interest due to their potential for medical and industrial applications.

Most proposals are inspired by bacteria with external flagella.

Could micro-robots driven by internal mechanisms be competitive? Synmov1 (Berg)
Thank you, Jerry!