MOMENTUM MAPS & CLASSICAL FIELDS

– Overview –

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Joint work over the years 1979–2009 with:

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Goals:

- Study the Lagrangian and Hamiltonian structures of classical field theories (CFTs) with constraints
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- Explore connections between initial value constraints & gauge transformations
- Tie together & understand many different and apparently unrelated facets of CFTs
- Focus on roles of gauge symmetry and momentum maps
Methods & Tools:

- calculus of variations
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- initial value analysis, Dirac constraint theory
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- multisymplectic geometry, multimomentum maps
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- Noether’s theorem
- energy-momentum maps
Generic Properties of Classical Field Theories

Gleaned from extensive study of standard examples:

- electromagnetism, Yang–Mills
- gravity
- strings
- relativistic fluids
- topological field theories . . .

Pioneers: Choquet-Bruhat, Lichnerowicz, Dirac–Bergmann, Arnowit–Deser–Misner (ADM), Fischer–Marsden...
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   — The corresponding **gauge group** is known at the outset.
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— The corresponding gauge group is known at the outset.

— Kinematic fields have no significance.

— Dynamic fields $\psi$, conjugate momenta $\rho$ have physical meaning.
2. The Euler–Lagrange equations are **overdetermined**: they include constraints

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— Elliptic system (typically)

— Assume all constraints are *first class* in the sense of Dirac
3. The $\Phi^i$ generate gauge transformations of $(\psi, \rho)$ via the canonical symplectic structure on the space of Cauchy data.
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— So the presence of constraints $\longleftrightarrow$ gauge freedom
4. The Hamiltonian (with respect to a slicing) has the form

\[ H = \int_\Sigma \sum_i \alpha_i \Phi^i(\psi, \rho) \, d\Sigma \]

depending linearly on the atlas fields \( \alpha_i \).
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Atlas fields:

- closely related to kinematic fields
- arbitrarily specifiable
- “drive” the entire gauge ambiguity of the CFT
5. *The evolution equations for the dynamic fields* \((\psi, \rho)\) *take the adjoint form*

\[
\frac{d}{d\lambda} \begin{pmatrix} \psi \\ \rho \end{pmatrix} = \mathbb{J} \cdot \sum_i \left[ D\Phi^i(\psi(\lambda), \rho(\lambda)) \right]^* \alpha_i. 
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— $\lambda$ is a slicing parameter ("time")

— $\mathbb{J}$ is a compatible almost complex structure; * an $L^2$-adjoint

— hyperbolic system (typically)
Adjoint form displays, in the clearest and most concise way, the interrelations between the

- dynamics
- initial value constraints, and
- gauge ambiguity of a theory
6. The Euler–Lagrange equations are equivalent, modulo gauge transformations, to the combined evolution equations (2) and constraint equations (1).
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— The constraints are preserved by the evolution equations
7. The space of solutions of the field equations is not necessarily smooth. It may have quadratic singularities occurring at symmetric solutions.
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— symplectic reduction

— linearization stability

— quantization
Philosophy

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— I wish to go further and provide (realistic) sufficient conditions which guarantee that they must occur in a CFT.

— I provide such criteria for 1–6 and lay the groundwork for 7.

A key objective is thus to derive the adjoint formalism for CFTs.
Example: (Abelian) Chern–Simons Theory

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- Gauge group is \( \text{Diff}(X) \): \( \eta \cdot A = \eta_* A \)
- E–L equations: \( dA = 0 \).
— Kinematic fields: $A_0$
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— Slicing of \( \Lambda^1(X) \rightarrow X \) generated by:

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\frac{d}{d\lambda} = \zeta^\mu \frac{\partial}{\partial x^\mu} - A_\nu \zeta^{\nu,\mu} \frac{\partial}{\partial A_\mu}
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— Gauge generators: $(D_i a) \frac{\delta}{\delta A^i}$
— Slicing of $\Lambda^1(X) \rightarrow X$ generated by:

$$\frac{d}{d\lambda} = \zeta^\mu \frac{\partial}{\partial x^\mu} - A_\nu \zeta^\nu,\mu \frac{\partial}{\partial A_\mu}$$

— Hamiltonian: $H = -2 \int_\Sigma (dA)_{12}(\zeta^\mu A_\mu) \, d\Sigma$
Atlas field: $\zeta^\mu A_\mu$

Evolution equations in adjoint form reduce to:
\[
\frac{d}{d\lambda} \left( A^i_\rho A^{\rho i} \right) = (D^i_\mu (\zeta^\mu A_\mu) \epsilon^{0ij} D^j_\nu (\zeta^\nu A_\nu))
\]
This is equivalent to
\[
(dA)^0_i = 0.
\]

N.B. We also have primary constraints $\rho^0 = 0$ and $\rho^i = \epsilon^{0ij} A^j$. 
— Atlas field: $\zeta^\mu A_\mu$

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$$\frac{d}{d\lambda} \begin{pmatrix} A_i \\ \rho^i \end{pmatrix} = \begin{pmatrix} D_i(\zeta^\mu A_\mu) \\ \epsilon^{0ij} D_j(\zeta^\mu A_\mu) \end{pmatrix}$$

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Traditional Approaches to CFT
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— Group-theoretical
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▶ concerned with the gauge covariance of a CFT
▶ Lagrangian-oriented
▶ covariant
▶ based on Noether’s theorem
— Canonical
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- initial value analysis
- Hamiltonian-oriented
- not covariant
- based on space + time decomposition
Connections:

— These two aspects of a mechanical system are linked by the momentum map.
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— One would like to have an analogous connection in CFT relating gauge symmetries to initial value constraints.
Caveat!

The standard notion of a momentum map associated to a symplectic group action usually cannot be carried over to spacetime covariant field theory, because:

— spacetime diffeomorphisms move Cauchy surfaces,
— the Hamiltonian formalism is only defined relative to a fixed Cauchy surface.
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Prime Example: Einstein’s theory of vacuum gravity

— the gauge group is the spacetime diffeomorphism group
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— the only remnants of this group on the instantaneous (i.e., space + time split) level are the superhamiltonian $\mathcal{H}$ and supermomenta $\mathcal{J}$

These deformations do not form a group, nor are $\mathcal{H}$ and $\mathcal{J}$ components of a momentum map.

This circumstance forces us to work on the covariant level.
Prime Example: Einstein’s theory of vacuum gravity

— the gauge group is the spacetime diffeomorphism group

— the only remnants of this group on the instantaneous (i.e., space + time split) level are the superhamiltonian $\mathcal{H}$ and supermomenta $\mathcal{J}$

  ▶ interpreted as the generators of temporal and spatial deformations of a Cauchy surface

  ▶ these deformations do not form a group

  ▶ nor are $\mathcal{H}$ and $\mathcal{J}$ components of a momentum map.

This circumstance forces us to work on the covariant level.
The Way Out: Multisymplectic Field Theory

We must construct a covariant counterpart to the instantaneous Hamiltonian formalism.

In the spacetime covariant (or multisymplectic) framework we develop here—an extension and refinement of the formalism of Kijowski and Szczyrba— the gauge group does act.

So we can define a covariant (or multi-) momentum map on the corresponding covariant (or multi-) phase space.
The Energy-Momentum Map

Key fact: The covariant momentum map induces an energy-momentum map $\Phi$ on the instantaneous phase space.
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— $\Phi$ is the crucial object reflecting the gauge transformation covariance of a CFT in the instantaneous picture.
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Key fact: The covariant momentum map induces an energy-momentum map $\Phi$ on the instantaneous phase space.

— Bridges the covariant & instantaneous formalisms

— $\Phi$ is the crucial object reflecting the gauge transformation covariance of a CFT in the instantaneous picture.

— In ADM gravity, $\Phi = -(\mathcal{H}, \mathcal{J})$, so that the superhamiltonian and supermomenta are the components of the energy-momentum map.
A recurrent theme is that the energy-momentum map encodes essentially all the dynamical information carried by a CFT: its

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- Hamiltonian (Item 4)
- gauge freedom (Item 3)
- initial value constraints (Item 2)
- stress-energy-momentum tensor
- ...
Indeed:

**Energy-Momentum Theorem**

The constraints (1) are given by the vanishing of the energy-momentum map associated to the gauge group of the theory.

Φ thus synthesizes the group-theoretical and canonical approaches to CFT.
Other Highlights

- Parametrization theory (à la Kuchař)
- Covariantization theory (à la Yang–Mills)
- Stress-energy-momentum tensors
- ‘Removing’ second class constraints (à la Stückelberg)