Riemannian cubics on Lie groups and orbits

David Meier, joint work with François Gay-Balmaz, Darryl Holm, Tudor Ratiu and François-Xavier Vialard

18 July
Have a Lie group $G$ (of **transformations**) acting transitively on a manifold $Q$ (the **object manifold**):

$$G \times Q \to Q, \quad (g, q) \mapsto gq.$$ 

Introduce a **right-invariant metric** $\gamma_G$ on $G$ and project the metric to $Q$ to obtain the **normal metric** $\gamma_Q$. 

**Occurs naturally:** Image/shape matching ($G = \text{Diff}$, $Q = \text{shapes}$); quantum mechanics ($G = \text{SU}(n+1)$, $Q = \text{CP}^n$ with Fubini-Study metric); unit sphere ($G = \text{SO}(3)$, $Q = S^2$ with usual metric) [Fields July 2012] Riemannian cubics 18 July 2 / 20
Plan of the talk (1)

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Occurs naturally:
- Image/shape matching ($G = \text{Diff}$, $Q =$ shapes);
- quantum mechanics ($G = SU(n+1)$, $Q = \mathbb{C}P^n$ with Fubini-Study metric);
- unit sphere ($G = SO(3)$, $Q = S^2$ with usual metric)
Consider a **higher-order variational principle** in both spaces. The solution curves are called **Riemannian cubics**. Generalization of cubic polynomials and cubic splines to Riemannian manifolds.

Interpolation problems where a certain degree of smoothness is required (piecewise geodesic interpolation leads to discontinuous velocities). This may be unnatural (shapes), or inconvenient (quantum control, camera configurations).
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Study the relationship between optimal curves on $G$ and optimal curves on $Q$, for the variational principle defined on the respective spaces.

Both variational principles (on $G$ or on $Q$) may be interesting in applications.

In geodesic shape matching one encounters geodesics on Diff with specific **form** of momenta/velocities (for example the singular momenta in landmark matching). One can explain this by understanding the relationship between geodesics on $G$ and geodesics on $Q$. 

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In geodesic shape matching one encounters geodesics on Diff with specific **form** of momenta/velocities (for example the singular momenta in landmark matching). One can explain this by understanding the relationship between geodesics on $G$ and geodesics on $Q$.

**Two main questions:**

- Which cubics on $Q$ can be lifted horizontally to cubics on $G$?
- Which cubics on $G$ project to cubics on $Q$? (more difficult)
Group actions and normal metrics

Actions

- Lie group $G$, object manifold $Q$.
- Transitive group action $G \times Q \to Q$, $(g, q) \mapsto gq$.
- Infinitesimal action $\xi_Q(q) := \partial_{\varepsilon=0} \exp(\varepsilon \xi)q$, where $\xi \in g$.

Metrics

- Right-invariant Riemannian metric $\gamma_G$ on $G$; $\gamma_G(u_g, v_g) = \gamma_e(\underline{u_g g^{-1}}, \underline{v_g g^{-1}})$, where $\gamma_e$ is the restriction of $\gamma_G$ to $T_e G \times T_e G$.
- Normal metric $\gamma_Q$ on $Q$ given by $\gamma_Q(u_q, u_q) = \inf\{\xi \in g | \xi_Q(q) = u_q\} \gamma_e(\xi, \xi)$.

Examples

- Lie group: $Q = G$, left multiplication; $\gamma_e$; obtain right-invariant metric.
- Unit sphere: $G = SO(3)$; $Q = S^2$; $\gamma_e(\Omega, \Omega) = \Omega \cdot \Omega$ (≈ 2-level QM system).
- Landmark matching: $G = \text{Diff}(\Omega)$; $Q = \mathbb{R}^3$; $\gamma_e$.
- Image matching: $G = \text{Diff}(\Omega)$; $Q = \mathcal{F}(\Omega)$; $\gamma_e$ inner product on $X(\Omega)$.
- Quantum mechanics: $G = SU(n+1)$; $Q = \mathbb{C}P^n$; $\gamma_e(A, B) = -2 \text{tr}(AB)$.
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- Will use raising/lowering bundle maps $\flat$ and $\sharp$ for both metrics. That is, $\flat : TG \to T^*G$ or $\flat : TQ \to T^*Q$; and $\sharp = \flat^{-1}$. 
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- Unit sphere: $G = SO(3)$; $Q = S^2$; $\gamma_e(\Omega, \Omega) = \Omega \cdot \Omega$ ($\cong$ 2-level QM system)
- Landmark matching: $G = \text{Diff}(\Omega)$; $Q = \mathbb{R}^3$; $\gamma_e = \ldots$
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Horizontality
Horizontal space

- Lie algebra of the isotropy subgroup of \( q \in Q \) denoted by \( \mathfrak{g}_q \) (vertical space)
- The horizontal space at \( q \) is \( \mathfrak{g}^\perp_q \).
- Horizontal projection \( H_q \) orthogonal projection onto \( \mathfrak{g}^\perp_q \).
- Normal metric can be written as \( \gamma_Q(u_q, u_q) = \gamma_e(H_q(\xi), H_q(\xi)) \) for any \( \xi \) with \( \xi_Q(q) = u_q \).
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Examples

- **Lie group**: $\mathfrak{g}_g^\perp = \mathfrak{g}$

- **Unit sphere**: $so(3)_x = \{\Omega \text{ with } \Omega \times x = 0\} \Rightarrow \{\Omega \text{ with } \Omega = \lambda x\}$. Therefore, $so(3)_x^\perp = x^\perp$. (2-dimensional)
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Momentum map

- Denote by $J : T^*Q \to \mathfrak{g}^*$ the **cotangent lift momentum map** for the action of $G$ and $Q$. Then $J^\#(\alpha_q) \in \mathfrak{g}_q^\perp$. Indeed,

$$\gamma_e(J^\#(\alpha_q), \xi) = \langle J(\alpha_q), \xi \rangle_{\mathfrak{g}^* \times \mathfrak{g}} = \langle \alpha_q, \xi_Q(q) \rangle_{T^*Q \times TQ}.$$  

- For $S^2$, $J^\#(p, x) = x \times p$. This is in $x^\perp = \mathfrak{so}(3)_x^\perp$. 

(Fields July 2012)  
Riemannian cubics  
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Fix a reference object $a \in Q$ and define the projection mapping

$$\Pi : G \to Q, \quad g \mapsto ga$$

This map is a Riemannian submersion. This means that vectors tangent to the object manifold $Q$ can be measured by lifting them horizontally to $TG$ and measuring the resulting horizontal vectors using $\gamma_G$.

- $V_g G = \ker T_g \Pi$ and $H_g G = V_g^\perp$ give orthogonal decomposition of $TG$ into horizontal and vertical subbundles

$$TG = HG \oplus VG.$$

- If $\Pi(g) = q$, then

$$V_g G = (g_q)g, \quad H_g G = (g_q^\perp)g.$$

- A curve $g(t) \in G$ is horizontal if $\dot{g} \in H_g G$. This is equivalent to $\xi = \dot{g}g^{-1} \in g_q^\perp$. 

(Fields July 2012) Riemannian cubics
Riemannian cubics

Denote by $D_t$ the covariant derivative of the Levi–Civita connection. In coordinates $(D_t \dot{q})^k = \ddot{q}^k + \Gamma^k_{ij} \dot{q}^i \dot{q}^j$.

Consider the second-order variational problem $\delta J = 0$ for

$$J[q] = \int_0^1 \|D_t \dot{q} \|^2 \, dt,$$

with respect to curves with fixed end-point velocities.

The Euler–Lagrange equation is [Noakes et al. 1989], [Crouch & Silva Leite 1995]

$$D_t^3 \dot{q}(t) + R(D_t \dot{q}(t), \dot{q}(t)) \dot{q}(t) = 0,$$

where $R$ is the curvature tensor $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Z$.

Solutions to this equation are called Riemannian cubics.
Cubics for normal metrics

**Goal:** Derive the equation for cubics in such a way that the horizontal generator of the curve appears. This will be helpful for analysing horizontal lifting properties.

- The **horizontal generator** of a curve $q(t) \in Q$ is the unique curve $\xi(t) \in g$ with $\dot{q} = \xi_Q(q)$ ("generator") and $\xi \in g_Q^\perp$ ("horizontal").

- Define the map $\bar{J} : TQ \to g$ by $\bar{J} := \sharp \circ J \circ b$, which "marries" $G$-action and metric. Then the horizontal generator of a curve $q(t) \in Q$ is given by $\bar{J}(\dot{q})$.

- Formula for the covariant derivative following from the horizontal lifting property of geodesics: $D_t\dot{q} = (\dot{J} + \text{ad}^{\dagger}_J \bar{J})_Q(q)$, where $\text{ad}^{\dagger}_{\eta} = \sharp \circ \text{ad}^*_\eta \circ b$.

- Fact: $\dot{J} + \text{ad}^{\dagger}_J \bar{J}$ is horizontal, that is, in $g_Q^\perp$.

\[ \Rightarrow \text{Rewrite the Lagrangian:} \]
\[
\|D_t\dot{q}\|_Q^2 = \|(\dot{J} + \text{ad}^{\dagger}_J \bar{J})_Q(q)\|_Q^2 \quad \text{normal metric} \quad \|\dot{J} + \text{ad}^{\dagger}_J\|_e^2
\]

\[ \Rightarrow \text{Euler–Lagrange equation:} \]
\[
\left[ \left( \frac{\delta \bar{J}}{\delta q} \right)^* - \frac{D}{Dt} \circ \left( \frac{\delta \bar{J}}{\delta \dot{q}} \right)^* \right] \left( \dot{\eta}^b + (\text{ad}_J \eta)^b - \text{ad}^*_\eta \bar{J}^b \right) = 0,
\]

where $\eta := \dot{J} + \text{ad}^{\dagger}_J \bar{J}$ and $\frac{\delta \bar{J}}{\delta q}, \frac{\delta \bar{J}}{\delta \dot{q}} : TQ \to g \Rightarrow \text{horizontal generator has appeared} \Rightarrow \text{Examples.}$
Cubics on Lie groups

A computation yields

$$
\left[ \left( \frac{\delta J}{\delta g} \right)^* - \frac{D}{Dt} \circ \left( \frac{\delta J}{\delta \dot{g}} \right)^* \right] \mu = (TR_{g^{-1}})^* \left( - \partial_t - \text{ad}^*_J \right) \mu
$$

for any curves $g(t) \in G$ and $\mu(t) \in \mathfrak{g}^*$. The Euler–Lagrange equation is therefore

$$
\left( \partial_t + \text{ad}^*_J \right) \left[ \dot{\eta}^b - (\text{ad}_J \eta)^b + \text{ad}^*_J \dot{J}^b \right] = 0,
$$

where $\eta := \dot{J} + \text{ad}^*_J \overline{J}$ and $\overline{J} := J(g, \dot{g}) = \dot{g}g^{-1}$. If $\gamma_G$ is bi-invariant this recovers the NHP equation [Noakes et al. 1989]

$$
\ddot{J} + [\ddot{J}, J] = 0.
$$

Alternative derivation on Lie groups proceeds via second-order Euler–Poincaré reduction: Lagrangian $L(g, \dot{g}) = \frac{1}{2} \| D_t \dot{g} \|^2 = \frac{1}{2} \| (\dot{J} + \text{ad}^*_J \overline{J})g \|^2$ is right-invariant with reduced Lagrangian $\ell(J, \dot{J}) = \frac{1}{2} \| \dot{J} + \text{ad}^*_J \overline{J} \|^2_e$. Second-order Euler–Poincaré equation is

$$
\left( \partial_t + \text{ad}^*_J \right) \left( \frac{\delta \ell}{\delta \overline{J}} - \partial_t \frac{\delta \ell}{\delta J} \right) = 0.
$$

For bi-invariance, $\ell(J, \dot{J}) = \frac{1}{2} \| \dot{J} \|^2_e \leadsto \text{NHP equation.}$
Assume that $\gamma_G$ is bi-invariant, and let $Q$ be a **symmetric space** for $G$. This means that $[g_q, g_q] \subset g_q$ for all $q \in Q$.

- The Euler–Lagrange equation is equivalent to $H_q(\dddot{\bar{J}} + 2[\ddot{J}, \bar{J}]) = 0$.
- In addition it is true for any curve $q(t) \in Q$ that $V_q(\dddot{\bar{J}} + 2[\ddot{J}, \bar{J}]) = 0$.
- **Therefore:** A curve $q(t) \in Q$ is a Riemannian cubic $\iff \dddot{\bar{J}} + 2[\ddot{J}, \bar{J}] = 0$.
  Derived in a different way in [Crouch & Silva Leite 1995].

- Recall that cubics on the group $G$ satisfy the NHP equation $\dddot{\bar{J}} + [\ddot{J}, \bar{J}] = 0$. 
Cubics on symmetric spaces

\[
\left[ \left( \frac{\delta \bar{J}}{\delta \bar{q}} \right)^* - \frac{D}{Dt} \circ \left( \frac{\delta \bar{J}}{\delta \bar{q}} \right)^* \right] \left( \bar{\eta}^b + (\text{ad} \bar{J} \eta)^b - \text{ad}^* \eta^b \bar{J}^b \right) = 0,
\]

Assume that \( \gamma_G \) is bi-invariant, and let \( Q \) be a **symmetric space** for \( G \). This means that \( [\mathfrak{g}_q^\perp, \mathfrak{g}_q^\perp] \subset \mathfrak{g}_q \) for all \( q \in Q \).

- The Euler–Lagrange equation is equivalent to \( H_q(\ddot{\bar{J}} + 2[\dot{\bar{J}}, \bar{J}]) = 0 \).
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- Recall that cubics on the group \( G \) satisfy the NHP equation \( \ddot{\bar{J}} + [\dot{\bar{J}}, \bar{J}] = 0 \).

\( \rightsquigarrow \) make use of similarity to analyse horizontal lifting properties.
Horizontal lifts of cubics on symmetric spaces

Goal: Find the cubics on $Q$ that can be lifted horizontally to cubics on $G$.

- Recall the Riemannian submersion $\Pi: G \to Q$, $g \mapsto ga$ with reference object $a$.
- Let $q(t)$ be a curve in $Q$ with $q(0) = a$. The curve defined by $g(0) = e$ and $\dot{g} = \bar{J}(\dot{q})g$ is horizontal above $q(t)$.
- Therefore, we are looking for the curves $q(t)$, which satisfy $\dddot{\bar{J}} + 2[\ddot{\bar{J}}, \dot{\bar{J}}] = 0$ (cubic on $Q$), and at the same time $\dddot{\bar{J}} + [\ddot{\bar{J}}, \dot{\bar{J}}] = 0$ (cubic on $G$).
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- Let $q(t)$ be a curve in $Q$ with $q(0) = a$. The curve defined by $g(0) = e$ and $\dot{g} = J(q)g$ is horizontal above $q(t)$.
- Therefore, we are looking for the curves $q(t)$, which satisfy $\ddot{J} + 2[\dot{J}, J] = 0$ (cubic on $Q$), and at the same time $\ddot{J} + [\dot{J}, J] = 0$ (cubic on $G$).

Theorem: A curve $q(t) \in Q$ is a Riemannian cubic and can be lifted horizontally to a Riemannian cubic $g(t) \in G$ if and only if it satisfies $\dot{q}(t) = (\xi(t))_Q(q(t))$ for a curve $\xi(t) \in g$ of the form

$$\xi(t) = \frac{ut^2}{2} + vt + w,$$

where $u, v, w$ span an Abelian subalgebra that lies in $g_{q(0)}^\perp$.

Proof: $\Rightarrow$ $\ddot{J}$ solves $\dddot{J} + [\dot{J}, J] = 0$ $\implies$ it follows (SSP) that $[\dot{J}, \ddot{J}] = 0$ $\implies$ from NHP equation: $\dddot{J} = \text{constant}$ $\implies$ $\dddot{J}$ is 2nd order polynomial in $t$. Coefficients mutually commuting since $[\dot{J}, \dddot{J}] = [\dddot{J}, \dot{J}] = 0$, and horizontal since $\dot{J}$ and $\dddot{J}$ as well as $\ddot{J}$ are horizontal (SSP).

$\leftarrow$ Assume $q(0) = a$. Start by showing that $\xi(t)$ is horizontal at all times. This makes use of $\operatorname{Exp}(\text{span}(u, v, w))$ being an Abelian subgroup, and bi-invariance of $\gamma_G$. $\implies$ Curve $g(0) = e$ and $\dot{g} = \xi g$ horizontal lift of $q(t)$. Both $g(t)$ and $q(t)$ are cubics.
Horizontal lifts of cubics on symmetric spaces (cont’d)

**Theorem:** A curve \( q(t) \in Q \) is a Riemannian cubic and can be lifted horizontally to a Riemannian cubic \( g(t) \in G \) if and only if it satisfies \( \dot{q}(t) = (\xi(t))_{Q}(q(t)) \) for a curve \( \xi(t) \in g \) of the form

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\xi(t) = \frac{ut^2}{2} + vt + w,
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where \( u, v, w \) span an Abelian subalgebra that lies in \( g_{q(0)} \).

- The **rank** of a symmetric space is the dimension of the maximal Abelian Lie subalgebra of \( g_{q}^{\perp} \). \( \Leftrightarrow \) The bigger the rank, the more vectors are compatible with the Theorem.

- In rank-one symmetric spaces \( u, v, w \) are all collinear.

**Corollary:** In rank-one symmetric spaces (\( S^2 \), for example) the only cubics that can be lifted horizontally to cubics are geodesics composed with a cubic polynomial in time.

**Proof:** Integrate \( \dot{q} = \left( \frac{at^2}{2} + bt + c \right) dQ(q) \). Find \( q(t) = e^{\left( \frac{at^3}{6} + \frac{bt^2}{2} + ct \right) d} q(0) \). ■
Horizontal lifts of cubics on symmetric spaces (cont’d)

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$\leadsto$ include non-horizontal curves on $G$. 

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**Projections of non-horizontal geodesics**

Stay in the symmetric space context for now. That is, $\gamma_G$ bi-invariant and $[g_q^+, g_q^+] \subset g_q$.

**Question:** Which non-horizontal **geodesics** on $G$ project to cubics on $Q$?

- First, describe geodesics on $G$. Euler–Poincaré equation is $\dot{\xi} = 0$, with reconstruction relation $\dot{g} = \xi g$.

- Let $q(t)$ be the projected curve $q(t) = \Pi(g(t)) = g(t)a$. Decompose $\xi$ into horizontal and vertical parts

  $$\xi = H_q(\xi) + V_q(\xi) = \bar{J}(\dot{q}) + \bar{\sigma}.$$ 

  Here we defined $\bar{\sigma} := V_q(\xi)$.

- Evolution equations are

  $$\dot{\bar{J}} = [\bar{\sigma}, \bar{J}], \quad \dot{\bar{\sigma}} = [\bar{J}, \bar{\sigma}].$$

  \(\leadsto\) have rewritten the geodesic equation on $G$.

- Recall that in order for $q(t)$ to be a cubic $\dddot{\bar{J}} + 2[\ddot{\bar{J}}, \bar{J}] = 0$ must hold.

**Theorem:** The projection $q(t)$ of a geodesic $g(t)$ is a Riemannian cubic if and only if at time $t = 0$

$$[\bar{\sigma}, [\bar{\sigma}, [\bar{\sigma}, \bar{J}]]] + [\bar{J}, [\bar{J}, [\bar{J}, \bar{\sigma}]]] = 0.$$
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$$\left[\bar{\sigma}, \left[\bar{\sigma}, [\bar{\sigma}, \bar{J}]\right]\right] + \left[\bar{J}, \left[\bar{J}, [\bar{J}, \bar{\sigma}]\right]\right] = 0.$$

Proof: Assume $q(0) = a$. We have $\bar{J}(t) = \text{Ad}_{g(t)} \bar{J}(0)$ and $\bar{\sigma}(t) = \text{Ad}_{g(t)} \bar{\sigma}(0) \rightsquigarrow$ if true at $t = 0$, then true at all times. Plugging in the geodesic equation into the equation for cubics one finds

$$\dddot{\bar{J}} + 2[\dddot{\bar{J}}, \bar{J}] = \left[\bar{\sigma}, \left[\bar{\sigma}, [\bar{\sigma}, \bar{J}]\right]\right] + \left[\bar{J}, \left[\bar{J}, [\bar{J}, \bar{\sigma}]\right]\right].$$

Special cases:

- $\bar{\sigma} = 0 \rightsquigarrow g(t)$ is horizontal geodesic, $q(t)$ is a geodesic.
- $[\bar{\sigma}, \bar{J}] = 0 \rightsquigarrow q(t)$ is a geodesic due to $D_t \dot{q} = \dot{J}_Q(q) = ([\bar{\sigma}, \bar{J}]_Q(q) = 0$.
- $[\bar{J}, [\bar{J}, \bar{\sigma}]] = c\bar{\sigma}$, $[\bar{\sigma}, [\bar{\sigma}, \bar{J}]] = c\bar{J}$, for $c \in \mathbb{R}$. 
Projections of non-horizontal geodesics (cont’d)

Consider $G = SO(3)$ and $Q = S^2$. A curve $x(t) \in S^2$ is generated by a rotation vector $\Omega = \vec{J} + \vec{\sigma}$. That is, $\dot{x} = \Omega \times x$. The theorem is equivalent to

$$\left(\|\vec{\sigma}\|^2 - \|\vec{J}\|^2\right) \vec{J} \times \vec{\sigma}.$$  

- $\vec{J} = 0$ or $\vec{\sigma} = 0 \Rightarrow$ trivial projected curves $x(t) = x(0)$, or projections of horizontal geodesics.
- $\|\vec{\sigma}\|^2 = \|\vec{J}\|^2 \Rightarrow$ for given initial velocity $\dot{x} = \mathbf{v}$, the projection $x(t)$ describes a constant speed rotation of $x(0)$ around the axis

$$\Omega = \vec{J} + \vec{\sigma} = x \times \dot{x} \pm \|\dot{x}\| x.$$
Include all cubics: Finding the obstruction term

*Goal:* Find the obstruction for the projection of a cubic to be again a cubic. We will use second-order Lagrange–Poincaré reduction for this [Cendra, Marsden, Ratiu 2001], [Gay-Balmaz, Holm, Ratiu 2011]. We still assume that $\gamma_G$ is bi-invariant, but we drop the condition $[g_q^+, g_q^+] \subset g_q$.

- Fix reference object $a \in Q$. Stabilizer $G_a$, with Lie algebra $g_a$. $\rightsquigarrow$ will reduce by $G_a$.
- Right-action of $G_a$,
  \[ \psi : G \times G_a \to G, \quad (g, h) = gh. \]
- Quotient space $G/G_a$ is diffeomorphic to $Q$. Recall the projection map $\Pi : G \to Q$, $g \mapsto ga$.
  \[ \rightsquigarrow \text{Principal bundle } (G, Q, G_a, \Pi, \psi). \]
- Introduce $g_a$-valued principal connection $A$,
  \[ [A] : TG \to g_a, \quad v_g \mapsto A_g(v_g) := V_a(g^{-1}v_g) \]
- Adjoint bundle is the associated vector bundle $[\tilde{g}_a] := (G \times g_a)/G_a$, where the quotient is taken wrt right-action of $G_a$ on $G \times g_a$,
  \[ (G \times g_a) \times G_a \to G \times g_a, \quad (g, \xi, h) \mapsto (gh, \text{Ad}_h^{-1} \xi). \]
- Induced linear connection with covariant derivative $D^A_t$
- Need map $i : \tilde{g}_a \to g$, $[g, \xi] \mapsto \text{Ad}_g \xi$, shorthand $\sigma \mapsto \bar{\sigma}$
Start by reviewing the geodesic case.

- First-order Lagrange–Poincaré reduction makes use of the bundle diffeomorphism
  \[ \alpha^{(1)}_A : TG/G_a \to TQ \times Q \tilde{g}_a, \quad [g, \dot{g}] \mapsto (q, \dot{q}) \times [g, A(\dot{g})]. \]

- Reduced variables \( q, \dot{q}, \sigma \).

- Geodesics on \( G \) arise as solutions to the kinetic energy action principle, where
  \[ L(g, \dot{g}) = \frac{1}{2} \| \dot{g} \|_G^2. \]
  \( \leadsto \) Reduced Lagrangian \( \ell(q, \dot{q}, \sigma) = \frac{1}{2} \| \dot{q} \|_Q^2 + \frac{1}{2} \| \sigma \|_{\tilde{g}_a}^2. \)

- Lagrange–Poincaré equations describing geodesics are
  \[ D_t \dot{q} = \nabla_{\dot{q}} \bar{\sigma}_Q, \quad D^A_t \sigma = 0. \]
  \( \leadsto \) this reveals the obstruction term for \( q(t) \) to be a geodesic.
Include all cubics: Finding the obstruction term (cont’d)

Find the obstruction for a cubic to project to a cubic.

- Second-order Lagrange–Poincaré reduction uses bundle diffeomorphism 
  \( \alpha^{(2)}_A : T^{(2)}G/G_a \rightarrow T^{(2)}Q \times Q 2\tilde{g}_a \),

\[
[g, \dot{g}, \ddot{g}] \mapsto (q, \dot{q}, \ddot{q}) \times [g, A(\dot{g})] \oplus D_t^A [g, A(\dot{g})] .
\]

- Reduced variables \( q, \dot{q}, \ddot{q}, \sigma, \dot{\sigma} \)

- Lagrangian of Riemannian cubics is 
  \( L(g, \dot{g}, \ddot{g}) = \frac{1}{2} \| D_t \dot{g} \|^2_G \)

\( \cong \) reduced Lagrangian

\[
\ell(q, \dot{q}, \ddot{q}, \sigma, \dot{\sigma}) = \frac{1}{2} \| D_t \dot{q} - \nabla_q \bar{\sigma} Q \|^2_Q + \frac{1}{2} \| \dot{\sigma} \|^2_{\tilde{g}_a}
\]

- The Lagrange–Poincaré equations are

\[
\boxed{D^3_t \dot{q} + R(D_t \dot{q}, \dot{q}) \dot{q}} = D^2_t \nabla_q \bar{\sigma} Q - \nabla \bar{\sigma}_Q^T \cdot D_t V_q - \nabla (\partial_t \bar{\sigma})_Q^T \cdot V_q + R(D_t \dot{q}, \bar{\sigma} Q(q)) \dot{q} + R(\nabla_q \bar{\sigma} Q, \dot{q} - \bar{\sigma} Q(q)) \dot{q} + \nabla_q \left( i \left( \bar{\sigma} + \text{ad}^\dagger \sigma \dot{\sigma} + i_q^T \partial_t \tilde{J}(V_q) \right) \right)_Q + F^T_\sigma \left( \mathcal{F}^\nabla \left( V^b_q, \dot{q} \right) \right)^\# \\
\left( D_t^A + \text{ad}^\dagger_\sigma \right) \left( \bar{\sigma} + i_q^T \partial_t \tilde{J}(V_q) \right) = 0.
\]
Thank you