Dirac structures

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Geometry, mechanics and dynamics: the legacy of J. Marsden
Fields Institute, July 2012
Outline:

1. Mechanics and constraints (Dirac’s theory)
2. “Degenerate” symplectic geometry: two viewpoints
3. Origins of Dirac structures
4. Properties of Dirac manifolds
5. Recent developments and applications
1. Phase spaces and constraints

- Symplectic phase space with constraint submanifold $C \hookrightarrow M$
  
  First class (coisotropic), second class (symplectic)...
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Questions:
Intrinsic geometry of constraints in Poisson phase spaces?
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Questions:

Intrinsic geometry of constraints in Poisson phase spaces?
Global structure behind “presymplectic foliations”? 
2. Two viewpoints to symplectic geometry

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Going degenerate: presymplectic and Poisson geometries...
3. Origins of Dirac structures

T. Courant’s thesis (1990):
Unified approach to presymplectic / Poisson structures
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Unified approach to presymplectic / Poisson structures

**Dirac structure:** subbundle $L \subset TM = TM \oplus T^*M$ such that

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Courant bracket on $\Gamma(\mathbb{T}M)$:

\[ \left[ (X, \alpha), (Y, \beta) \right] = \left[ [X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\beta(X) - \alpha(Y)) \right] . \]
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Non-skew bracket: $[(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X\beta - i_Yd\alpha)$. 
Examples

*initial examples...*
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Another example...

\[ M = \mathbb{R}^3, \quad \text{coordinates } (x, y, z) \]

\[ L = \text{span}\left\langle \left(\frac{\partial}{\partial y}, zd\right), \left(\frac{\partial}{\partial x}, -zd\right), (0, dz) \right\rangle \]
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\( L = \text{span}\left\langle \left( \frac{\partial}{\partial y}, zdx \right), \left( \frac{\partial}{\partial x}, -zdy \right), (0, dz) \right\rangle \)

For \( z \neq 0 \), this is graph of \( \pi = \frac{1}{z} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} : \)

\[ \{x, y\} = \frac{1}{z}, \quad \{x, z\} = 0, \quad \{y, z\} = 0. \]

\textit{singular Poisson versus smooth Dirac ...}
4. Properties of Dirac manifolds

- Lie algebroid...
- Presymplectic foliation
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Quotient Poisson manifolds...

Dirac structures = “pre-Poisson”
Inducing Dirac structures on submanifolds

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◊ Smoothness issue

Try pulling back \( \pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \) to \( x \)-axis...
Inducing Dirac structures on submanifolds

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◊ Smoothness issue
Try pulling back \( \pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \) to \( x \)-axis...

◊ Transversality condition:
Enough that \( L \cap TC^\circ \) has constant rank.
Poisson-Dirac submanifolds of Poisson manifolds \((M, \pi)\). Pull-back of \(\pi\) to \(C\) is smooth and Poisson \((TC \cap \pi^\#(TC^o) = 0)\).
Poisson-Dirac submanifolds of Poisson manifolds $(M, \pi)$. Pull-back of $\pi$ to $C$ is smooth and Poisson ($\mathcal{T}C \cap \pi^\#(\mathcal{T}C^o) = 0$) "Leafwise symplectic submanifolds": generalizes symplectic submanifolds to Poisson world...
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◊ **Moment level sets**

$J : M \to g^*$ Poisson map (=moment map), $C = J^{-1}(0) \hookrightarrow M$

Transversality ok e.g. if 0 is regular value, $g$-action free.

Moment level set inherits Dirac structure.
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**Dirac geometry** = intrinsic geometry of constraints...
5. Recent developments and applications

- Courant algebroids, twist by closed 3-forms
- Lie algebroids/groupoids, equivariant cohomology
- Generalized symmetries and moment maps (e.g. $G$-valued ...)
- Spinors and generalized complex geometry
- Supergeometric viewpoint

Back to mechanics:

- Lagrangian systems with constraints (nonholonomic), implicit Hamiltonian systems (e.g. electric circuits); generalizations to field theory (multi-Dirac)...
- Geometry of nonholonomic brackets...

among others...
Twists by closed 3-forms
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$\phi$-twisted Courant bracket:

$$[(X, \alpha), (Y, \beta)]_{\phi} = [(X, \alpha), (Y, \beta)] + i_Y i_X \phi.$$
Twists by closed 3-forms

Consider closed 3-form $\phi \in \Omega^3_{cl}(M)$:

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Then

- Dirac structures: modified integrability conditions, but similar properties...
- Twisted Poisson structure: $\frac{1}{2}[\pi, \pi] = \pi^\#(\phi)$
The Cartan-Dirac structure on Lie groups

$G$ Lie group, $\langle \cdot, \cdot \rangle_g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ Ad-invariant.
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\( G \) Lie group, \( \langle \cdot, \cdot \rangle_g : g \times g \to \mathbb{R} \) Ad-invariant.

Cartan-Dirac structure:

\[
L_G := \{ (u^r - u^l, \frac{1}{2} \langle u^r + u^l, \cdot \rangle_g ) \mid u \in g \}.
\]

This is \( \phi_G \)-integrable, where \( \phi_G \in \Omega^3(M) \) is the Cartan 3-form.
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**Singular foliation:** Conjugacy classes

**Leafwise 2-form** (G.H.J.W. '97):

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\omega(u_G, v_G)|_{g} := \left\langle \frac{\text{Ad}_g - \text{Ad}_{g^{-1}}}{2} u, v \right\rangle_g
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Compare with Lie-Poisson on \( g^* \)...
Supergeometric viewpoint
### Supergeometric viewpoint

| \( (E, \langle \cdot, \cdot \rangle) \) | \( (\mathcal{M}, \{\cdot, \cdot\}) \) deg. 2, symplectic N-manifold |
| \( [\cdot, \cdot] \), \( \rho \) | \( \Theta \in C_3(\mathcal{M}), \{\Theta, \Theta\} = 0 \) |
| \( L \subset E, \ L = L^\perp \) | \( \mathcal{L} \subset \mathcal{M} \) Lagrangian submanifold |
| Dirac structure \( L, [\Gamma(L), \Gamma(L)] \subseteq \Gamma(L) \) | Lagrangian submf. \( \mathcal{L}, \Theta|_\mathcal{L} \equiv \text{cont.} \) |
Supergeometric viewpoint

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After all, everything is a Lagrangian submanifold...