A remark on lifting problem of KK-elements between dimension drop algebras

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Motivations:

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- Such a lifting problem is closely related to the classification of C*-algebras: when the approximate (asymptotic) unitary equivalence classes of homomorphisms are determined by their induced KK-classes, for the corresponding existence theorem, we need to lift a KK-class to a homomorphism.

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- For classification, try to connect the criterion above to an invariant for C*-algebras, for example, an order structure on the K-groups.

Cuntz's picture of KK-groups

Definition

For two C*-algebras A and B, define $KK(A, B)$ to be the homotopy classes of quasi-homomorphisms from A to B, where a quasi-homomorphism is a pair of homomorphisms $\phi_{\pm} : A \to M(B \otimes \mathcal{K})$ with $\phi_{+}(a) - \phi_{-}(a) \in B \otimes \mathcal{K}$.

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- Circle algebras: from UCT, a KK-group of two such algebras A and B is $Hom(K_0(A), K_0(B)) \oplus Hom(K_1(A), K_1(B))$. The order structure on K-groups is introduced by Elliott, and we can look at this as follows: regard $K_*(A) = K_0(A) \oplus K_1(A)$ as $KK(C(S^1), A)$, then $K_*^+(A) \triangleq \{([\varphi(1)], [\varphi(e^{2\pi it})]) \,|\, \varphi$ is a homomorphism from $C(S^1)$ to $M_k(A)$ for some $k\}$.

Examples (continued): what else should we look at? Torsion K_1 group.
Dimension drop algebra I_n and $\widetilde{I_n}$,

$$I_n = \{f : [0,1] \to M_n \,|\, f(0) = 0, f(1) = \lambda 1, \lambda \in \mathbb{C}\}.$$

For UCT, this time, we have a nontrivial Ext. part, so situation is not as same as before. Indeed,

$$K_*^+(\widetilde{I_n}) = \{(a, \bar{b}) \,|\, a \geq 1\} \cup (0,0).$$

There exists a KK-element $[\delta_1] - [\delta_0]$, which can not be lifted to a homomorphism.

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$K_0(A;\mathbb{Z}/p\mathbb{Z}) \triangleq K_1(A\otimes P) = KK(P,A)$ for any C*-algebra P in the Bootstrap class such that $K_0(P)=0$ and $K_1(P)=\mathbb{Z}/p\mathbb{Z}$. $K_0(A;\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}) \triangleq KK(\widetilde{P},A)$. We can choose P to be I_p.

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Definition

The order structure is defined as follows:

$$K_0^+(A;\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}) \triangleq \{([\varphi(1)],[\varphi|_{I_p}])\,|\,\varphi\in Hom(\widetilde{I_p},M_k(A))\}.$$

Lemma

There is a natural short exact sequence of groups:

$$K_0(A) \stackrel{\times p}{\rightarrow} K_0(A) \stackrel{\mu_{A;p}}{\rightarrow} K_0(A;\mathbb{Z}_p) \stackrel{\nu_{A;p}}{\rightarrow} K_1(A) \stackrel{\times p}{\rightarrow} K_1(A).$$

where $p \geq 2$, $\mu_{A;p}, \nu_{A;p}$ *are the Bockstein operations defined by the Kasparov product with the element of* $KK(I_p, \mathbb{C})$ *given by the evaluation* $\delta_1 : I_p \to \mathbb{C}$ *and the element of* $KK^1(\mathbb{C}, I_p)$ *given by the inclusion* $i : SM_p \to I_p$ *respectively.*

Lemma

For any KK-element $\alpha \in KK(A,B)$, then α induces the following commutative diagram:

$$\begin{array}{ccccccccc}
K_0(A) & \xrightarrow{\times p} & K_0(A) & \xrightarrow{\mu_{A;p}} & K_0(A;\mathbb{Z}_p) & \xrightarrow{\nu_{A;p}} & K_1(A) & \xrightarrow{\times p} & K_1(A) \\
 & & \downarrow{K_0(\alpha)} & & \downarrow{K_0(\alpha;\mathbb{Z}_p)} & & \downarrow{K_1(\alpha)} & & \\
K_0(B) & \xrightarrow{\times p} & K_0(B) & \xrightarrow{\mu_{B;p}} & K_0(B;\mathbb{Z}_p) & \xrightarrow{\nu_{B;p}} & K_1(B) & \xrightarrow{\times p} & K_1(B)
\end{array}.$$

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Theorem

Given n, m, For any p, with n divides p, if $\alpha \in KK(\widetilde{I_n}, \widetilde{I_m})$ induces a positive homomorphism on the K-groups with coefficient above, then α can be lifted to a homomorphism.

Generalized dimension drop algebras

Jiang and Su investigated the following dimension drop algebras:

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$\mathbf{I}[m_0, m, m_1]$, is of the form:

$$\mathbf{I}[m_0, m, m_1] = \{f \in \mathrm{C}([0,1], \mathrm{M}_m) : f(0) = a_0 \otimes \mathrm{id}_{\frac{m}{m_0}}, f(1) = \mathrm{id}_{\frac{m}{m_1}} \otimes a_1\},$$

where a_0 and a_1 belong to $\mathrm{M}_{m_0}(\mathbb{C})$ and $\mathrm{M}_{m_1}(\mathbb{C})$ respectively.

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Answer: No.

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Moreover, they proved that $KK(A, B) \cong Hom(K^0(B), K^0(A))$ under the Kasparov product. This picture is quite good.

Examples: for the dimension drop algebra above, say $(m_0, m_1) = 1$, look at the induced triple on K-theory with coefficient, we have the following:

Theorem

Suppose the K_1 multiplicity is zero, then every triple Γ is of the form:

$$\Gamma = (\beta_0 x - dm_1)K(\delta_0; p) + (\beta_1 x + dm_0)K(\delta_1; p).$$

where (β_0, β_1) is an auxiliary pair with $\beta_0 \geq 0, \beta_1 \leq 0$, such that $\beta_0 m_0 + \beta_1 m_1 = 1$.

Theorem

For the triple Γ, under some conditions which involve some data above, it preserves the Dadarlat-Loring order on K-theory coefficient if and only if the first part coefficient above is positive.

Then, combine with Jiang and Su's criterion, we can find examples of KK-elements which preserve Dadarlat-Loring order but fail to be lifted to a homomorphism.