The Russo-Dye Theorem
in Nest Subalgebras of factors

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Russo and Dye (1966) given a $\mathcal{C}^*$-algebra $\mathcal{A}$, the closure of the convex hull of all the unitary elements in $\mathcal{A}$ is the unit ball of $\mathcal{A}$. 
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A nest $\mathcal{N}$ is a chain of subspaces of a separable Hilbert space $\mathcal{H}$ containing $(0)$ and $\mathcal{H}$ which is closed under intersection and closed span.
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Define the immediate successor of $N \in \mathcal{N}$

$$N_+ = \inf\{M \in \mathcal{N} : M > N\}$$

and the immediate predecessor of $N$

$$N_- = \sup\{M \in \mathcal{N} : M < N\}.$$
The nest algebra $\text{Alg}\mathcal{N}$ is the set of all the operators $T$ in $B(\mathcal{H})$ such that $TN \subseteq N$ for every $N$ in $\mathcal{N}$. 
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Let $\mathcal{H} = \mathbb{C}^n$, $\{e_1, e_2, \cdots, e_n\}$ the orthogonal basis for $\mathcal{H}$. Let $N_k = \text{span}\{e_1, e_2, \cdots, e_k\}$, then the nest algebra $\mathcal{T}_n$ is all the $n \times n$ upper triangular matrices.
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Let \( \mathcal{H} = \mathbb{C}^n \), \( \{e_1, e_2, \cdots, e_n\} \) the orthogonal basis for \( \mathcal{H} \). Let \( N_k = \text{span}\{e_1, e_2, \cdots, e_k\} \), then the nest algebra \( \mathcal{T}_n \) is all the \( n \times n \) upper triangular matrices.

Russo-Dye Theorem is not true for this case. Every unitary element in \( \mathcal{T}_n \) must be diagonal, and so the closure of the convex hull of all the unitary elements is smaller than the whole unit ball of \( \mathcal{T}_n \).
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Davidson (1998) the weak operator topology closure of the convex hull of all the unitary elements in a nest algebra is the whole unit ball of the nest algebra if and only if the nest is admissible in $B(\mathcal{H})$.

A nest is said to be admissible in $B(\mathcal{H})$, if both $0_+$ and $I-I_-$ are either zero or infinite rank.
If $\mathcal{N}$ is an admissible nest in $B(\mathcal{H})$, then there is a projection $P = \sum_{k \in \mathbb{Z}} E_k$ in the nest algebra, which is called an admissible projection for $\mathcal{N}$. 
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The one dimensional projections are equivalent in the Murry-von Neumann sense.
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Nest subalgebras of von Neumann algebras were first introduced by R. I. Loebl and P. S. Muhly. They showed that nest subalgebras of von Neumann algebras are precisely the algebras of analytic operators with respect to certain one parameter groups of inner \(*\)-automorphisms of the von Neumann algebras.
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Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N} \subseteq \mathcal{M}$ be a nest, then the nest subalgebra of the von Neumann algebra is the set of all the operators $T$ in $\mathcal{M}$ such that $TN \subseteq N$ for every $N$ in $\mathcal{N}$, which is denoted by $\mathcal{M} \cap \text{Alg}\mathcal{N}$. 
Definition

Let $\mathcal{N}$ be a nest in a factor $\mathcal{R}$. $\mathcal{N}$ is said to be an admissible nest in $\mathcal{R}$, if for any proper projection $N \in \mathcal{N}$, i.e., $N \neq 0$ or $I$, both $N$ and $I - N$ are infinite projections in $\mathcal{R}$. 
**Definition**

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If $\mathcal{M}$ is a type $I$ factor, then $\mathcal{M}$ is $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. It is easy to verify that this definition is a generalization of that in $B(\mathcal{H})$. 
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Lemma

Let $\mathcal{N}$ be an admissible nest in a factor $\mathcal{R}$ with $0_+ > 0$ and $I_- = I$. Then there exist two projection sequences:

$\{N_k \in \mathcal{N} : k = 0, 1, 2, \cdots \}$ increasing to $I$ with

$$0 = N_0 < 0_+ = N_1 < N_2 < N_3 < \cdots < I$$

and $\{E_k \in \mathcal{R} \cap \text{Alg}\mathcal{N} : k \in \mathbb{Z}\}$ with $E_i \sim E_j$, for $i, j \in \mathbb{Z}$, such that

$$\begin{cases} E_k < 0_+ , & k = 0, -1, -2, \cdots . \\ E_k \leq N_{k+1} - N_k , & k = 1, 2, \cdots . \end{cases}$$
Lemma

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with $E_i \sim E_j$, for $i, j \in \mathbb{Z}$, such that

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E_k < 0_+, & k = 0, -1, -2, \cdots \\
E_k \leq N_{k+1} - N_k, & k = 1, 2, \cdots
\end{cases}
\]

We call $\mathcal{N} = \{I, N_k : k = 0, 1, 2, \cdots \}$ and $\{E_k : k \in \mathbb{Z}\}$ the basic sub-nest of $\mathcal{N}$ and the basic equivalent projection sequence for $\mathcal{N}$, respectively.
The Russo-Dye Theorem

Motivation

Factors

General von Neumann Algebras

Theorem

Let $\mathcal{N}$ be an admissible nest in a factor $\mathcal{R}$, then each element $A \in \mathcal{R} \cap \text{Alg}\mathcal{N}$ with $\|A\| < 1 - \frac{1}{n}$ is the average of $16n^2$ unitary elements in $\mathcal{R} \cap \text{Alg}\mathcal{N}$. Thus the convex hull of all the unitary elements contains the open unit ball of $\mathcal{R} \cap \text{Alg}\mathcal{N}$. If $\mathcal{N}$ is not admissible, then the weak operator topology closure of the convex hull of all the unitary elements in $\mathcal{R} \cap \text{Alg}\mathcal{N}$ is not the unit ball.
Sketch of the proof: Suppose \( \{E_k\} \) is the basic equivalent projection sequence. Let \( P = \sum_{k=\infty}^{+\infty} E_{2k} \) and \( Q = \sum_{k=-\infty}^{+\infty} E_{2k+1} \). Split \( P \) and \( Q \) into \( 2n \) orthogonal projections \( P_1, P_2, \ldots, P_{2n} \) and \( Q_1, Q_2, \ldots, Q_{2n} \), respectively. I.e.,

\[
P = \sum_{k=1}^{2n} P_k \quad \text{and} \quad Q = \sum_{k=1}^{2n} Q_k.
\]

It follows that \( X_i = Q_i \downarrow - \frac{1}{2n} Q_\downarrow \) and \( Y_j = P_j \downarrow - \frac{1}{2n} P_\downarrow \) are contractions such that

\[
\sum_{i=1}^{2n} X_i = (2n - 1)I = \sum_{j=1}^{2n} Y_j.
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\[
\sum_{i=1}^{2n} X_i = (2n - 1)I = \sum_{j=1}^{2n} Y_j.
\]

Moreover, \((1 - \frac{1}{2n})^2 > 1 - \frac{1}{n}\). Thus

\[
A_{ij} = \left(\frac{2n}{2n - 1}\right)^2 X_i A Y_j = Q_i^\perp A_{ij} P_j^\perp, \text{ for } 1 \leq i, j \leq n
\]

are strict contractions in \( \mathcal{R} \cap Alg\mathcal{N} \) and \( A = \frac{1}{4n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij} \).
Dilate $A_{ij} = Q_i^\perp A_{ij} P_j^\perp$ to be a unitary element $U_{ij}$ in $\mathcal{R} \cap \text{AlgN}$ such that $A_{ij} = Q_i^\perp U_{ij} P_j^\perp$. 
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$$\frac{1}{4} \sum_{k=1}^{4} U_{ijk} = R^\perp U S^\perp = A_{ij}.$$
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That is:

$$\frac{1}{16n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} \sum_{k=1}^{4} U_{ijk} = \frac{1}{4n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij} = A.$$
Definition

Let $\mathcal{N}$ be a nest in a von Neumann algebra $\mathcal{M}$. $\mathcal{N}$ is said to be an admissible nest in $\mathcal{M}$, if for each projection $N \in \mathcal{N}$, both $N - N_c$ and $I - N - (I - N)_c$ are either 0 or proper infinite in $\mathcal{M}$. Here $N_c$ is the union of all the central projections $P_a$ of $\mathcal{M}$ such that $P_a \leq N$. $N_c$ is the maximal central projection such that $N_c \leq P_a$. In factors, all the central projections are trivial and a projection is infinite if and only if it is proper infinite.
Definition

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Here $N_c$ is the union of all the central projections $P_a$ of $\mathcal{M}$ such that $P_a \leq N$. $N_c$ is the maximal central projection such that $N_c \leq P$.

In factors, all the central projections are trivial and a projection is infinite if and only if it is proper infinite.
For every von Neumann algebra $\mathcal{M}$, there are orthogonal center projections $P_{I_n}, 1 \leq n < \infty, P_{I_\infty}, P_{II_1}, P_{II_\infty}$ and $P_{III}$ with sum $I$ and $P_{I_\infty}\mathcal{M}, P_{II_1}\mathcal{M}, P_{II_\infty}\mathcal{M}$ and $P_{III}\mathcal{M}$ are type $I_n$, type $I_\infty$, type $II_1$, type $II_\infty$ and type $III$ von Neumann algebras, respectively.
For every von Neumann algebra $\mathcal{M}$, there are orthogonal center projections $P_{I_n}$, $1 \leq n < \infty$, $P_{I_{\infty}}$, $P_{II_1}$, $P_{II_{\infty}}$ and $P_{III}$ with sum $I$ and $P_{I_{\infty}}\mathcal{M}$, $P_{II_1}\mathcal{M}$, $P_{II_{\infty}}\mathcal{M}$ and $P_{III}\mathcal{M}$ are type $I_n$, type $I_{\infty}$, type $II_1$, type $II_{\infty}$ and type $III$ von Neumann algebras, respectively.

If $\mathcal{N}$ is an admissible nest in $\mathcal{M}$, then each of the summands is decomposable

$$P_A\mathcal{M} = \sum P^j_A\mathcal{M}, \quad A = I_n, I_{\infty}, II_1, II_{\infty}, III$$

$P^j_A$'s are orthogonal central projections and each $P^j_A\mathcal{N}$ in $P^j_A\mathcal{M}$ is either a trivial nest or

$$C_{P^j_A\mathcal{N}} = C_{P^j_A(I-N)} = P^j_A, \quad N \neq 0, I.$$
For every von Neumann algebra $\mathcal{M}$, there are orthogonal center projections $P_{I_n}$, $1 \leq n < \infty$, $P_{I\infty}$, $P_{II_1}$, $P_{II\infty}$ and $P_{III}$ with sum $I$ and $P_{I\infty}\mathcal{M}$, $P_{II_1}\mathcal{M}$, $P_{II\infty}\mathcal{M}$ and $P_{III}\mathcal{M}$ are type $I_n$, type $I\infty$, type $II_1$, type $II\infty$ and type $III$ von Neumann algebras, respectively.

If $\mathcal{N}$ is an admissible nest in $\mathcal{M}$, then each of the summands is decomposable

$$P_A\mathcal{M} = \sum P_A^j\mathcal{M}, \quad A = I_n, I\infty, II_1, II\infty, III$$

$P_A^j$'s are orthogonal central projections and each $P_A^j\mathcal{N}$ in $P_A^j\mathcal{M}$ is either a trivial nest or

$$C_{P_A^j\mathcal{N}} = C_{P_A^j(I-N)} = P_A, \quad N \neq 0, I.$$

The nests $P_A^j\mathcal{N}$, $A = I_n, II_1$ are trivial in $P_A^j\mathcal{M}$. 
Type $III$ case, use central carrier, Type $II_\infty$ and Type $I_\infty$ case, use tracial weight on each factor, we can obtain the basic sub-nest $\{N_t\}$ and basic equivalent projection sequence $\{E_n\}$ for each summand of $P_A^j \chi_{Y_n} M$, $A = I_\infty, II_\infty, III$. 
The Russo-Dye Theorem

Let $\mathcal{N}$ be an admissible nest in a von Neumann $\mathcal{M}$ acting on a separable infinite dimensional Hilbert space $\mathcal{H}$, then each element $A \in \mathcal{M} \cap \text{Alg}\mathcal{N}$ with $\|A\| < 1 - \frac{1}{n}$ is the average of $16n^2$ unitary elements in $\mathcal{M} \cap \text{Alg}\mathcal{N}$. Thus the convex hull of all the unitary elements contains the whole open unit ball.

If $\mathcal{N}$ is not admissible, the weak operator topology closure of the convex hull of all the unitary elements in $\mathcal{M} \cap \text{Alg}\mathcal{N}$ is not the unit ball of $\mathcal{M} \cap \text{Alg}\mathcal{N}$. 
Sketch of the proof: If the nest \( N \) is not admissible in \( M \), then, without loss of generality, we may suppose \( N \in \mathcal{N} \) with \( N - N_c \neq 0 \) is infinite but not proper infinite.
Sketch of the proof: If the nest $\mathcal{N}$ is not admissible in $\mathcal{M}$, then, without loss of generality, we may suppose $N \in \mathcal{N}$ with $N - N_c \neq 0$ is infinite but not proper infinite. If $N - N_c$ is infinite but not proper infinite, then there is a central projection $Q$ in $\mathcal{M}$ such that $QN - QN_c \neq 0$ is finite in $\mathcal{M}$. For any unitary element $U \in \mathcal{M} \cap Alg\mathcal{N}$, we have 

$$(QN - QN_c)U(QN - QN_c)^\perp = 0.$$
Sketch of the proof: If the nest $\mathcal{N}$ is not admissible in $\mathcal{M}$, then, without loss of generality, we may suppose $N \in \mathcal{N}$ with $N - N_c \neq 0$ is infinite but not proper infinite. If $N - N_c$ is infinite but not proper infinite, then there is a central projection $Q$ in $\mathcal{M}$ such that $QN - QN_c \neq 0$ is finite in $\mathcal{M}$. For any unitary element $U \in \mathcal{M} \cap \text{Alg}\mathcal{N}$, we have $(QN - QN_c)U(QN - QN_c)^\perp = 0$. This means

$$C_{(QN - QN_c)}C_{(QN - QN_c)}^\perp = 0.$$ 

We have $C_{QN - QN_c} = QN - QN_c$ and $C_{(QN - QN_c)}^\perp = (QN - QN_c)^\perp$. Therefore $QN - QN_c \neq 0$ is a central projections. But $QN_c = (QN)_c$ is the maximal central projection such that $QN_c \leq QN$, a contradiction.
Thank You!