Closed Unitary and Similarity Orbits of Normal Operators in Purely Infinite C^*-Algebras

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Background

Definition

Let $\mathfrak{A}$ be a unital C*-algebra. Two operators $A, B \in \mathfrak{A}$ are said to be approximately unitarily equivalent in $\mathfrak{A}$, denoted $A \sim_{au} B$, if there exists a sequence of unitary operators $(U_n)_{n\geq 1} \subseteq \mathfrak{A}$ such that

$$\lim_{n\to\infty} \|A - U_n B U_n^*\| = 0.$$

Theorem (Weyl-von Neumann-Berg Theorem)

Let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be normal operators on a separable Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. $N_1 \sim_{au} N_2$.
2. $rank(\chi_U(N_1)) = rank(\chi_U(N_2))$ *for all open subsets* U *of* $\mathbb{C}$.
3. $\sigma(N_1) = \sigma(N_2)$ *and* $dim(ker(\lambda I_{\mathcal{H}} - N_1)) = dim(ker(\lambda I_{\mathcal{H}} - N_2))$ *for all isolated* $\lambda \in \sigma(N_1)$.

Background - BDF

Theorem (Brown-Douglas-Fillmore; 1973)

Let N_1 and N_2 be normal operators in the Calkin algebra. Then the following are equivalent:

1. $N_1 \sim_{au} N_2$.
2. $N_1 = UN_2U^*$ *for some unitary operator* U *in the Calkin algebra.*
3. $\sigma(N_1) = \sigma(N_2)$ *and the Fredholm index of* $\lambda I - N_1$ *and* $\lambda I - N_2$ *agree for all* $\lambda \notin \sigma(N_1)$.

Background - Dadarlat's Result

Theorem (Dadarlat; 1995)

Let X be a compact metric space, let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, and let $\varphi, \psi : C(X) \to \mathfrak{A}$ be two unital, injective *-homomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if $[[\varphi]] = [[\psi]]$ in $KL(C(X), \mathfrak{A})$.

Corollary (Specific Case of Dadarlat's Result)

Let N_1 and N_2 be two normal operators in a unital, simple, purely infinite C^*-algebra. Then $N_1 \sim_{au} N_2$ if and only if

1. $\sigma(N_1) = \sigma(N_2)$,
2. $[\lambda I_{\mathfrak{A}} - N_1]_1 = [\lambda I_{\mathfrak{A}} - N_2]_1$ *for all* $\lambda \notin \sigma(N_1)$, *and*
3. N_1 *and* N_2 *have equivalent common spectral projections.*

Approximate Unitary Equivalence - Traditional Techniques

Theorem (Lin; 1996)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N \in \mathfrak{A}$ be a normal operator. Then N can be approximated by normal operators with finite spectrum if and only if $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$.

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Theorem (Alternate Proof)

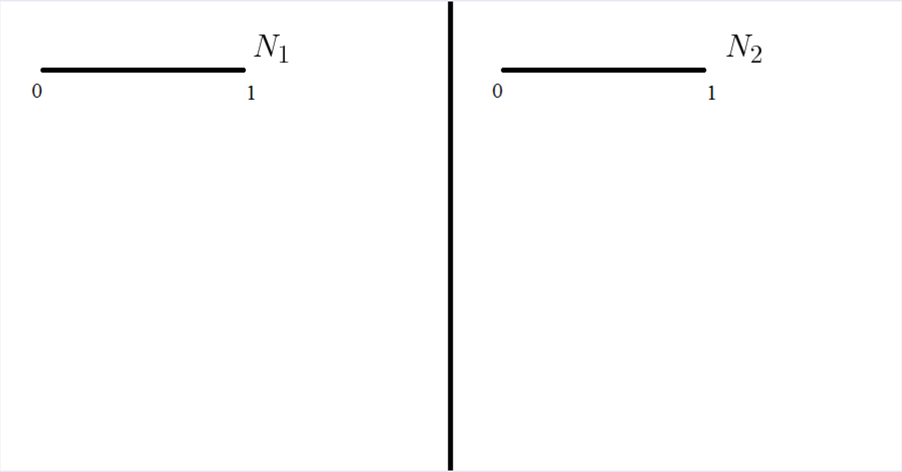
Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. If

1. $\sigma(N_1) = \sigma(N_2)$,
2. $\lambda I_{\mathfrak{A}} - N_1, \lambda I_{\mathfrak{A}} - N_2 \in \mathfrak{A}_0^{-1}$ *for all* $\lambda \notin \sigma(N_1)$, *and*
3. N_1 *and* N_2 *have equivalent common spectral projections,*

then $N_1 \sim_{au} N_2$.

Sketch of Proof.

Suppose $\sigma(N_1) = \sigma(N_2) = [0, 1]$.



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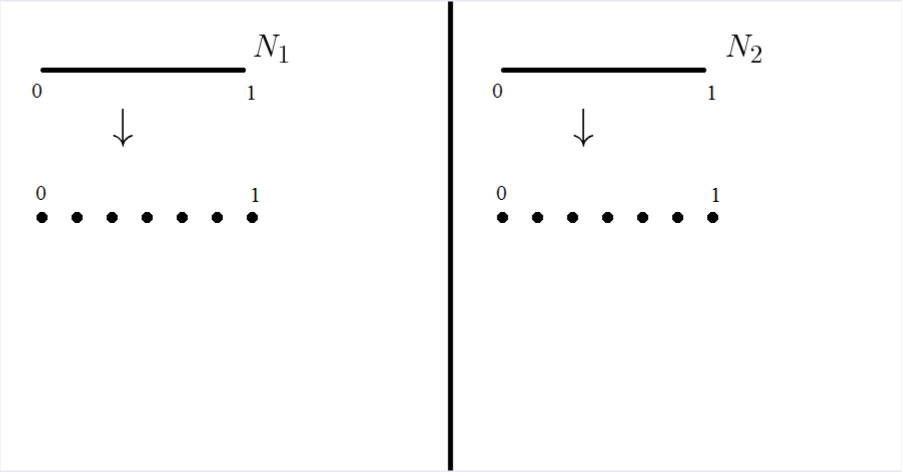

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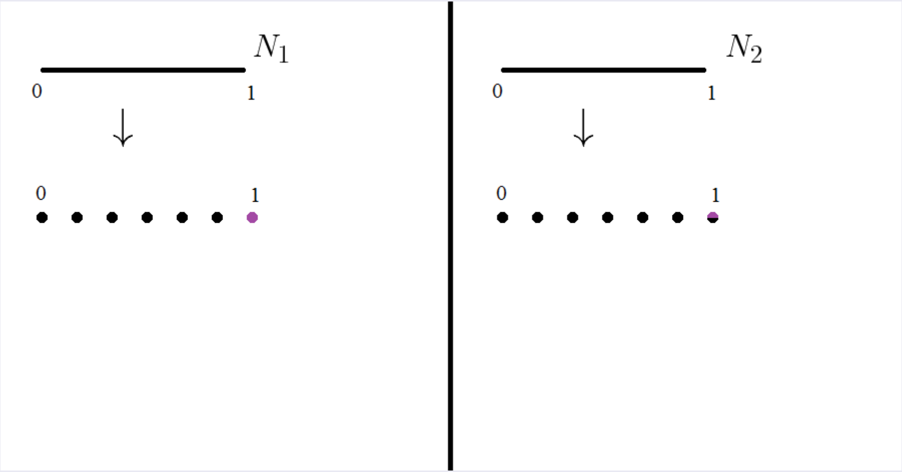

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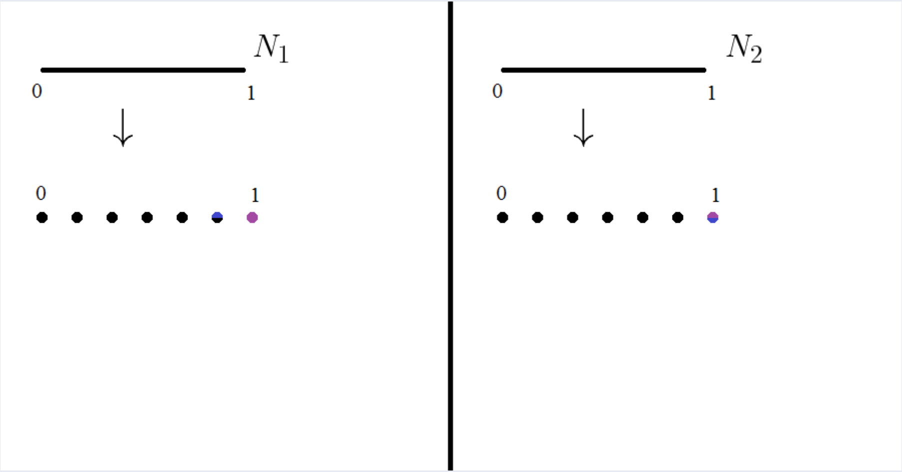

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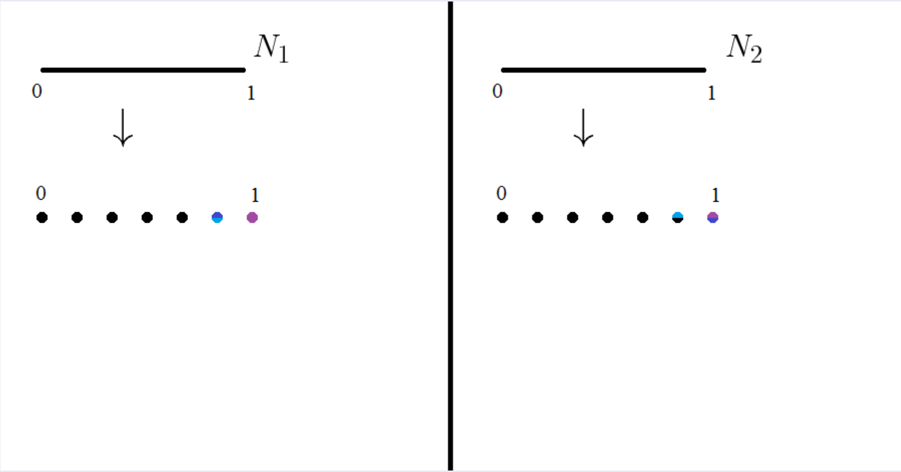

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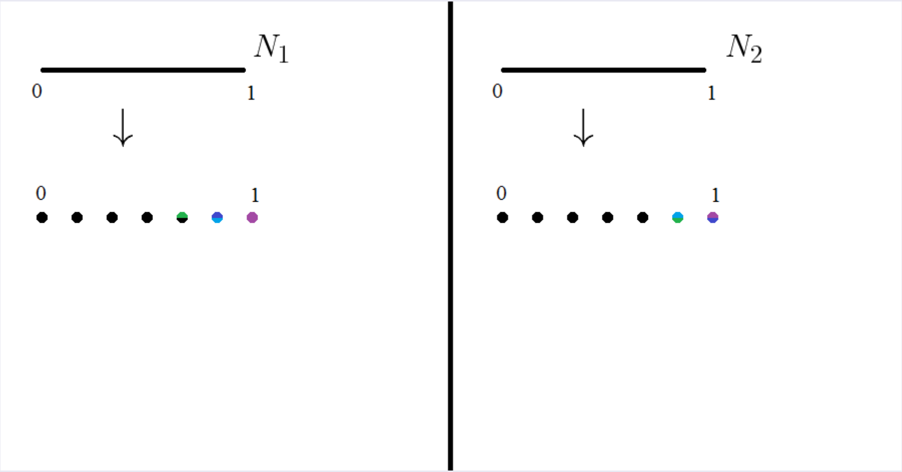

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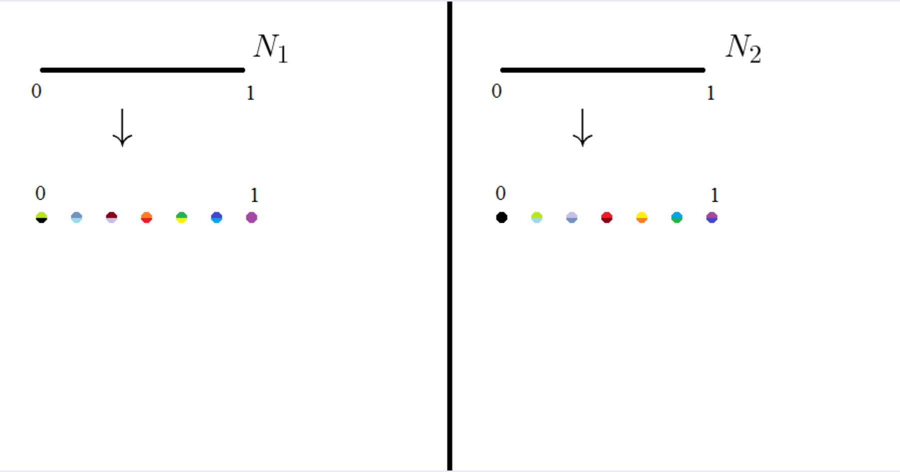

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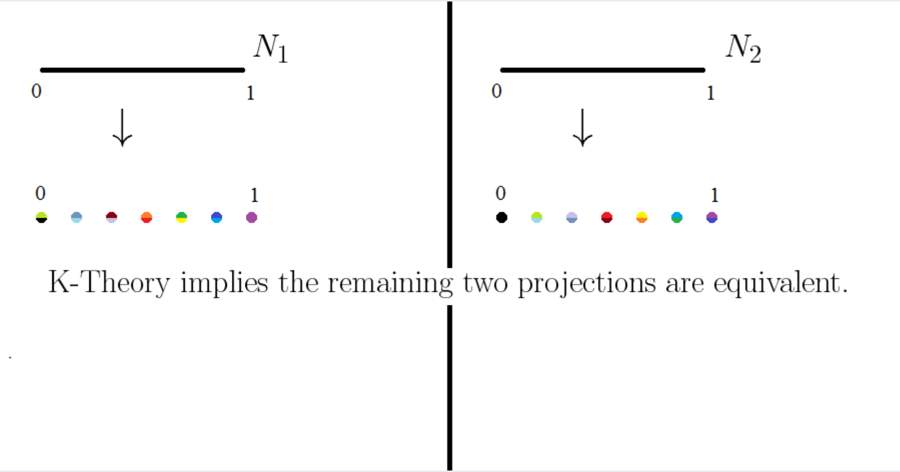

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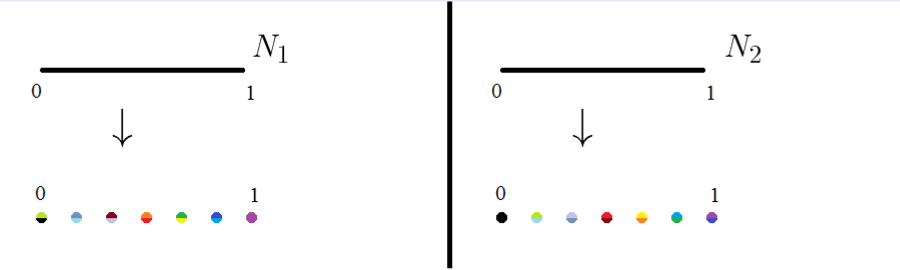

K-Theory implies the remaining two projections are equivalent.

$$U := \sum_{j=0}^{n} V_j + \sum_{j=0}^{n-1} W_j$$

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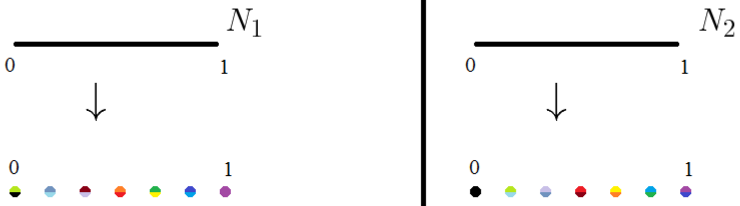

K-Theory implies the remaining two projections are equivalent.

$$U := \sum_{j=0}^{n} V_j + \sum_{j=0}^{n-1} W_j$$

$$\|N_1 - U^* N_2 U\| \leq \epsilon$$

Distance Between Unitary Orbits - Definitions

Definition

Let $\mathfrak{A}$ be a unital C*-algebra and let $\mathcal{U}(\mathfrak{A})$ denote the group of unitaries. For an operator $A, B \in \mathfrak{A}$, the unitary orbit of A in $\mathfrak{A}$ is the set

$$\mathcal{U}(A) := \{UAU^* \mid U \in \mathcal{U}(\mathfrak{A})\}.$$

The distance between the unitary orbits of A and B is

$$\begin{aligned} dist(\mathcal{U}(A), \mathcal{U}(B)) &= \inf\{\|A' - B'\| \mid A' \in \mathcal{U}(A), B' \in \mathcal{U}(B)\} \\ &= \inf\{\|A - UBU^*\| \mid U \in \mathcal{U}(\mathfrak{A})\}. \end{aligned}$$

Distance Between Unitary Orbits - $\mathcal{B}(\mathcal{H})$

Definition

Let X and Y be subsets of $\mathbb{C}$. The Hausdorff distance between X and Y, denoted $d_H(X, Y)$, is

$$d_H(X, Y) := \max\left\{\sup_{x\in X} dist(x, Y), \sup_{y\in Y} dist(y, X)\right\}.$$

Theorem (Davidson; 1986)

Let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be normal operators. Then

$$dist(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq d_H(\sigma(N_1), \sigma(N_2)).$$

If $\mathcal{H}$ is infinite dimensional and separable and if $\sigma(N_j) = \sigma_e(N_j)$, then the above is an equality.

Distance Between Unitary Orbits - Calkin Algebra

Definition

Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. For normal operators $N_1, N_2 \in \mathfrak{A}$ let $\rho(N_1, N_2)$ denote the maximum of $d_H(\sigma(N_1), \sigma(N_2))$ and

$$\sup \left\{ dist(\lambda, \sigma(N_1)) + dist(\lambda, \sigma(N_2)) \;\middle|\; \begin{array}{c} \lambda \notin \sigma(N_1) \cup \sigma(N_2) \\ [\lambda I_{\mathfrak{A}} - N_1]_1 \neq [\lambda I_{\mathfrak{A}} - N_2]_1 \end{array} \right\}.$$

Theorem (Davidson; 1984)

If N_1 and N_2 are normal operators in the Calkin algebra, then

$$\rho(N_1, N_2) \leq dist(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 2\rho(N_1, N_2).$$

Distance Between Unitary Orbits - U.S.PI.

Generalizing Davidson's proof gives:

Theorem

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Then

$$dist(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq \rho(N_1, N_2).$$

Distance Between Unitary Orbits - Trivial Index Function

Theorem (Skoufranis; 2013)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_j \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_j)$. If N_1 and N_2 have equivalent common spectral projections, then

$$dist(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H\left(\sigma(N_1), \sigma(N_2)\right).$$

Distance Between Unitary Orbits - A Construction

Lemma (Skoufranis; 2013)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra, let $V \in \mathfrak{A}$ be a non-unitary isometry, and let $P := VV^*$. Then there exists a unital embedding of the 2^∞-UHF C^*-algebra $\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ into $(I_\mathfrak{A} - P)\mathfrak{A}(I_\mathfrak{A} - P)$ such that $[Q]_0 = 0$ in $\mathfrak{A}$ for every projection $Q \in \mathfrak{B}$.

Distance Between Unitary Orbits - Additional Bounds

Theorem (Skoufranis; 2013)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra. If $N_1, N_2 \in \mathfrak{A}$ are normal operators with equivalent common spectral projections, then

$$dist(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 2\rho(N_1, N_2).$$

Theorem (Skoufranis; 2013)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that

1. $[\lambda I_{\mathfrak{A}} - N_1]_1 = [\lambda I_{\mathfrak{A}} - N_2]_1$ *for all* $\lambda \notin \sigma(N_1) \cup \sigma(N_2)$, *and*
2. N_1 *and* N_2 *have equivalent common spectral projections.*

Then

$$dist(\mathcal{U}(N), \mathcal{U}(M)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Closed Similarity Orbits - Definitions

Definition

Let $\mathfrak{A}$ be a unital C*-algebra. The similarity orbit of an operator $A \in \mathfrak{A}$ is the set

$$\mathcal{S}(A) := \left\{ VAV^{-1} \mid V \in \mathfrak{A}^{-1} \right\}.$$

The closed similarity orbit of A is $\overline{\mathcal{S}(A)}$ (the norm closure).

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The closed similarity orbit of A is $\overline{S(A)}$ (the norm closure).

Theorem (Barría and Herrero; 1978)

If $\mathcal{H}$ is a separable Hilbert space and $N, M \in \mathcal{B}(\mathcal{H})$ are normal operators with $\sigma(M) = \sigma_e(M)$, $N \in \overline{\mathcal{S}(M)}$ if and only if

1. $\sigma(M) \subseteq \sigma(N)$ *and* $\sigma_e(M) \subseteq \sigma_e(N)$,
2. *if $\lambda \in \sigma(N)$ is isolated, $\ker(\lambda I_{\mathcal{H}} - M)$ and $\ker(\lambda I_{\mathcal{H}} - N)$ have the same dimension, and*
3. *if $\lambda \in \sigma_e(N)$ is not isolated in $\sigma_e(N)$, the component of λ in $\sigma_e(N)$ contains some non-isolated point of $\sigma_e(M)$.*

Closed Similarity Orbits - Calkin Algebra

Theorem (Apostol, Herrero, Voiculesu; 1982)

Let N and M be normal operators in the Calkin algebra. Then $N \in \overline{\mathcal{S}(M)}$ if and only if

1. $\sigma_e(M) \subseteq \sigma_e(N)$,
2. *each component of* $\sigma_e(N)$ *intersects* $\sigma_e(M)$,
3. *the Fredholm index of* $\lambda I - M$ *and* $\lambda I - N$ *agree for all* $\lambda \notin \sigma_e(N)$, *and*
4. *if* $\lambda \in \sigma_e(N)$ *is not isolated in* $\sigma_e(N)$, *the component of* λ *in* $\sigma_e(N)$ *contains some non-isolated point of* $\sigma_e(M)$.

Closed Similarity Orbits - Purely Infinite C*-Algebras

Theorem (Skoufranis; 2013)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N, M \in \mathfrak{A}$ be normal operators. Then $N \in \overline{\mathcal{S}(M)}$ if and only if

1. $\sigma(M) \subseteq \sigma(N)$,
2. *each component of $\sigma(N)$ intersects $\sigma(M)$,*
3. $[\lambda I_{\mathfrak{A}} - N]_1 = [\lambda I_{\mathfrak{A}} - M]_1$ *for all* $\lambda \notin \sigma(N)$,
4. *if $\lambda \in \sigma(N)$ is not isolated in $\sigma(N)$, the component of λ in $\sigma(N)$ contains some non-isolated point of $\sigma(M)$, and*
5. *N and M have equivalent common spectral projections.*

Closed Similarity Orbits - Elements of the Proof

Theorem (Marcoux, Skoufranis; 2012)

Let $\mathfrak{B} := \overline{\bigcup_{n\geq 1} \mathcal{M}_{2^n}(\mathbb{C})}$ *be the* 2^∞*-UHF* C^**-algebra. Then there exists a normal operator* $N \in \mathfrak{B}$ *with* $\sigma(N) = \overline{\mathbb{D}}$ *such that* N *is a norm limit of nilpotent matrices from* $\bigcup_{n\geq 1} \mathcal{M}_{2^n}(\mathbb{C})$*.*

Lemma

Let $\mathfrak{A}$ *be a unital, simple, purely infinite* C^**-algebra, let* $M \in \mathfrak{A}$ *be a normal operator, let* $V \in \mathfrak{A}$ *be a non-unitary isometry, let* $P := VV^*$*, and let* $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ *be the unital copy of the* 2^∞*-UHF* C^**-algebra as constructed before. Suppose* μ *is a cluster point of* $\sigma(M)$ *and* $Q \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$ *is a nilpotent matrix for some* $\ell \in \mathbb{N}$*. Then* $VMV^* + \mu(I_{\mathfrak{A}} - P) + Q \in \overline{\mathcal{S}(M)}$*.*

Closed Similarity Orbits - Limits of Nilpotent Operators

Theorem (Skoufranis; 2013)

Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N \in \mathfrak{A}$ be a normal operator. Then N is a limit of nilpotent operators if and only if $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ whenever $\lambda \notin \sigma(N)$.

Thanks for Listening!