Nonself-adjoint 2-graph Algebras

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Row-isometries

Let $S_1, \ldots, S_n$ be isometries on $\mathcal{H}$ with pairwise orthogonal ranges, i.e.

$$S_i^* S_j = \delta_{i,j} I.$$

Then $S = [S_1, \ldots, S_n]$ is a row-isometry, i.e. is an isometric map from $\mathcal{H}^{(n)}$ to $\mathcal{H}$.
Conversely an isometric map from $\mathcal{H}^{(n)}$ is determined by n isometries on $\mathcal{H}$ with pairwise orthogonal ranges.
We say a row-isometry is of *Cuntz-type* if

$$\sum_{i=1}^{n} S_i S_i^* = I.$$

We will be interested in "commuting" row-isometries and the algebras they generate.

Motivation: different algebras in the 1-graph case

Let $S = [S_1, \ldots, S_n]$ be a Cuntz-type row-isometry. Then

1. there is only one possible C^*-algebra (Cuntz),
2. there is only one possible unital norm-closed algebra (Popescu),
3. the weak operator closed unital nonself-adjoint algebras are determined by the structure of the row-isometry (Davidson-Katsoulis-Pitts; Kennedy).

Representations of single vertex 2-graphs

Let $S = [S_1, \ldots, S_m]$ and $T = [T_1, \ldots, T_n]$ be row-isometries on $\mathcal{H}$ and let θ be a permuation on $m \times n$ elements. Then S and T are θ-commuting row-isometries if

$$S_i T_j = T_{j'} S_{i'} \text{ when } \theta(i,j) = (i',j').$$

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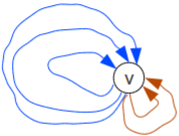
This is precisely saying that (S, T) is an isometric representation of the 2-graph



An important example: the left-regular representation

Let $\mathcal{H}_n = \ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w : \ w \in \mathbb{F}_n^+\}$. Define the row-isometry $L = [L_1, \ldots, L_n]$ by

$$L_i \xi_w = \xi_{iw}.$$

Let $\mathcal{A}_n = \overline{\mathrm{alg}}^{\|\cdot\|}\{I, L_1, \ldots, L_n\}$. We call this the *noncommutative disc algebra*. (Note when $n = 1$, $\mathcal{A}_1 = A(\mathbb{D})$).

Let $\mathfrak{L}_n = \overline{\mathrm{alg}}^{\mathrm{WOT}}\{I, L_1, \ldots, L_n\}$. We call this the *noncommutative analytic Toeplitz algebra*. (Note when $n = 1$, $\mathfrak{L}_1 = H^\infty$).

An important example: the left-regular representation

Let θ be a permutation on $m \times n$ and let $\mathbb{F}_\theta^+$ be the unital semigroup

$$\mathbb{F}_\theta^+ = \langle e_1, \dots, e_m, f_1, \dots, f_n : \; e_i f_j = f_{j'} e_{i'} \text{ when } \theta(i,j) = (i',j') \rangle.$$

Let $\mathcal{H}_\theta = \ell^2(\mathbb{F}_\theta^2)$ with orthonormal basis $\{\xi_w : \; w \in \mathbb{F}_\theta^+\}$. Define θ-commuting row-isometries $E = [E_1, \dots, E_m]$ and $F = [F_1, \dots, F_n]$ by

$$E_i \xi_w = \xi_{e_i w} \text{ and } F_j \xi_w = \xi_{f_j w}.$$

Let $\mathcal{A}_\theta = \overline{\text{alg}}^{\|\cdot\|}\{I, E_1, \dots, E_m, F_1, \dots, F_n\}$. We call this the *higher-rank noncommutative disc algebra*.
Let $\mathfrak{L}_\theta = \overline{\text{alg}}^{\text{WOT}}\{I, E_1, \dots, E_m, F_1, \dots, F_n\}$. We call this the *higher-rank noncommutative analytic Toeplitz algebra*

Nonself-adjoint 2-graph algebras

We will be primarily interested in θ-commuting row-isometries (S, T) where both S and T are Cuntz-type. These are precisely the Cuntz-Krieger families for the 2-graph $\mathbb{F}_\theta^+$.

Definition

Let (S, T) be a pair of θ-commuting Cuntz-type row-isometries. We call the algebra

$$\mathcal{S} = \overline{\text{alg}}^{\text{WOT}}\{I, S_1, \ldots, S_m, T_1, \ldots, T_n\}$$

a *nonself-adjoint 2-graph algebra.*

Definition

Let S be a row-isometry. We call the algebra

$$\mathcal{S} = \overline{\text{alg}}^{\text{WOT}}\{I, S_1, \ldots, S_m\}$$

a *free semigroup algebra.*

The Structure of Free semigroup algebras

Theorem (Davidson, Katsoulis & Pitts (2001))

Let S be a row-isometry on $\mathcal{H}$. Let $\mathcal{S}$ be the unital weakly closed algebra generated by S and let $\mathcal{M}$ be the von-Neumann algebra generated by S.
Then there is a projection P in $\mathcal{S}$ so that

1. *$P^\perp \mathcal{H}$ is an invariant subspace for $\mathcal{S}$,*
2. $\mathcal{S} = \mathcal{M}P + P^\perp \mathcal{S} P^\perp$,
3. *$P^\perp \mathcal{S} P^\perp$ is "like" $\mathfrak{L}_n$.*

The Structure of nonself-adjoint 2-graphs

Theorem (F. & Yang (2013))

Let (S, T) be Cuntz-type θ-commuting row-isometries on $\mathcal{H}$. Let $\mathcal{S}$ be the nonself-adjoint 2-graph generated by S and T and let $\mathcal{M}$ be the von-Neumann algebra generated by S and T.
Then there is a projection P in $\mathcal{S}$ so that

1. *$P^\perp \mathcal{H}$ is an invariant subspace for $\mathcal{S}$,*
2. *$\mathcal{S} = \mathcal{M}P + P^\perp \mathcal{S} P^\perp$.*

The Structure projection

Let (S, T) be a Cuntz-type representation of $\mathbb{F}_\theta^+$ and let $\mathcal{S}$ be the nonself-adjoint 2-graph algebra generated by (S, T).
Note that $[S_1T_1, S_1T_2, \ldots, S_mT_n]$ is a row-isometry.

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For any $k, l \geq 0$ we have a row-isometry

$$[ST]_{k,l} := [S_wT_u : \ |w| = k, \ |u| = l].$$

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Each of these row-isometries generates a free semigroup algebra in side $\mathcal{S}$. Let $\mathcal{S}_{k,l}$ be the free semigroup algebra generated by $[ST]_{k,l}$.
By Davidson-Katsoulis-Pitts each $\mathcal{S}_{k,l}$ has a structure projection $P_{k,l}$. Then

$$P = \bigwedge_{k,l>0} P_{k,l}.$$

What about the bottom corner?

Question

In our structure theorem above, there was no description of what the corner $P^{\perp}\mathcal{S}P^{\perp}$ was like. Why not?

Answer

Our setting is too general.

Example

Let S be any Cuntz-type row-isometry and let $T = S$. Then (S, T) are θ-commuting row-isometries (for some θ). So the nonself-adjoint 2-graph generated by (S, T) is just the free semigroup algebra generated by S.

The above example is a representation of a periodic 2-graph.

Aperiodicity

Periodicity of 2-graphs is a technical condition about the existence of repetition in infinite red-blue paths. If (S, T) is a Cuntz-type representation of an aperiodic 2-graph then there will necessarily be a strong relation between S and T making them behave more like a 1-graph than a 2-graph.

Lemma (Davidson & Yang (2009))

Let (S, T) be θ-commuting Cuntz-type row-isometries where $\mathbb{F}_\theta^+$ is a periodic 2-graph. Then there are $a, b > 0$ such that $m^a = n^b$ such that

$$[S_v : |v| = a] = [T_u W : |u| = b],$$

where W is a unitary in the center of the C^*-algebra generated by S and T.

The Structure of nonself-adjoint 2-graphs

Theorem (F. & Yang (2013))

Let (S, T) be Cuntz-type θ-commuting row-isometries on $\mathcal{H}$. Let $\mathcal{S}$ be the nonself-adjoint 2-graph generated by S and T and let $\mathcal{M}$ be the von-Neumann algebra generated by S and T
Then there is a projection P in $\mathcal{S}$ so that

1. $P^\perp \mathcal{H}$ *is an invariant subspace for* $\mathcal{S}$,
2. $\mathcal{S} = \mathcal{M}P + P^\perp \mathcal{S} P^\perp$.

Further, if θ defines an aperodic 2-graph then there is a projection Q such that $Q \geq P^\perp$ and

3. $Q\mathcal{H}$ *is an invariant subspace for* $\mathcal{S}$,
4. $Q\mathcal{S}Q$ *is "like"* $\mathfrak{L}_\theta$.

Norm-closed algebras

Theorem (Popescu)

Let $S = [S_1, \ldots, S_n]$ be any row-isometry. Then

$$\mathcal{A} = \overline{alg}^{\|\cdot\|}\{I,\, S_1, S_2, \ldots, S_n\}$$

is completely isometrically isomorphic to the noncommutative disc algebra $\mathcal{A}_n$.

This does not hold for isometric representations of 2-graphs. Not even for aperiodic 2-graphs:

Example

Let $L = [L_1, \ldots, L_n]$ be the left regular representation of $\mathbb{F}_n^+$ and let $R = [R_1, \ldots, R_n]$ be the right regular representation. Then $L_i R_j = R_j L_i$. It can be shown that $\overline{\mathrm{alg}}^{\|\cdot\|}\{I,\, L_i,\, R_j\}$ is not completely isometrically isomorphic to $\mathcal{A}_{\mathrm{id}}$.

Rigidity

However, in our setting something similar to Popescu's result does hold:

Theorem (F. & Yang 2013)

Let (S, T) be an isometric representation of an aperiodic 2-graph $\mathbb{F}_\theta^+$ on a Hilbert space $\mathcal{H}$. Let
$\mathcal{A} = \overline{\mathrm{alg}}^{\|\cdot\|}\{I, S_1, \ldots, S_m, T_1, \ldots, T_n\}$.
Suppose there is a Cuntz-type representation (S', T') of $\mathbb{F}_\theta^+$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that (S, T) is the restriction of (S', T'), i.e. each $S_i = S_i'|_{\mathcal{H}}$ and $T_j = T_j'|_{\mathcal{H}}$.
Then $\mathcal{A}$ is completely isometrically isomorphic to $\mathcal{A}_\theta$.