COMBINATORIAL RIGIDITY AND GRAPH CONSTRUCTIONS

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1. Introduction

Rigidity theory has its origins in the work of Cauchy and Euler on convex polyhedra. It is a fascinating subject that draws on many areas of mathematics and has wide ranging applications in, for example, structural engineering and material science.

A *framework* (G, p) is the combination of a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. (G, p) is *rigid*, [1], if there is no edge length preserving continuous deformation of the vertices that is not a rigid motion of $\mathbb{R}^d$, i.e. is not derived from translations and/or rotations. Moreover (G, p) is minimally rigid if it is rigid but for all edges $e \in E$ the framework $(G - e, p)$ is not rigid (flexible).

Rigidity is a generic property in the sense that if there is one choice of p for which (G, p) is rigid then for almost all choices q, (G, q) is rigid. It is standard therefore to take an algebraic definition of a generic framework and then to refer to the abstract graph as rigid or flexible.

Combinatorially the main problem is to analyse classes of graphs determined by simple vertex/edge counting conditions. For example it is a fundamental result of Laman [6] that the class of $(2, 3)$-tight graphs are exactly the graphs with minimally rigid generic realisations in the plane.

A graph $G = (V, E)$ is $(2, 3)$-*sparse* if $|E(X)| \leq 2|V(X)| - 3$ for every subgraph X with $|E(X)| > 0$ and G is $(2, 3)$-*tight* if it is $(2, 3)$-sparse and $|E| = 2|V| - 3$.

The key step in proving Laman's theorem is to show that the Henneberg construction moves (see, for example, [13]) generate all $(2, 3)$-tight graphs from K_2. This is an attractive result because the idea of the Henneberg moves is easily understood, see Figure 1.

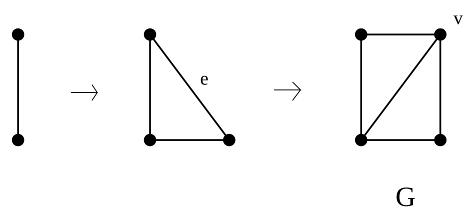

Figure 1. A recursive construction of a $(2, 3)$-tight graph G from the complete graph on two vertices, K_2, by applying a Henneberg 1 move and then a Henneberg 2 move.

Algorithmically this class of graphs is difficult to check directly but Recski [11] showed the equivalence with a variant of the well known spanning tree decomposition for $(2,2)$-tight graphs (see [14] and [8]). Such graphs can be verified in polynomial time. There are also natural Pebble game algorithms for such graphs [7].

More generally G is (k,l)-*sparse* if $|E(X)| \leq k|V(X)| - l$ and is (k,l)-*tight* if it is (k,l)-sparse and $|E| = k|V| - l$. The natural generalisation of Laman's theorem to 3-dimensions fails (see, for example, [3]). Although solutions for particular classes of graphs and of frameworks do exist, the relevant graphs, the $(3,6)$-tight graphs, are not fully understood.

A deeper related topic is the problem of global rigidity (when there is a unique arrangement of the vertices subject to the edge length constraints), here as for minimal rigidity there is a complete solution for generic frameworks in 2-dimensions, see Jackson and Jordan [5], whose generalisation fails in 3-dimensions. The approach in [5] uses the Henneberg construction moves and the concept of a connected rigidity matroid. (A matroid is a combinatorial structure generalising linear independence of vectors and the rigidity matroid is a particular example arising from the linear independence of the Jacobean derivative matrix of the system of edge (length) equations of a given framework.)

It is also natural to consider an analysis of the classes of $(2,l)$-tight graphs, for $l = 3,2,1,0$. This has been done from a variety of perspectives, see for example [6], [9], [10], [4] and [12].

By combining some of these ideas there are a number of open problems that are reasonably accessible. In particular the following are potential avenues of development.

(1) Henneberg-type recursive constructions for $(2,l)$-tight multigraphs.
(2) Algorithms for $(2,l)$-tight simple graphs.
(3) The set of $(2,2)$-tight simple graphs (together with K_2 and K_3) forms a matroid. The circuits of this matroid are $(2,1)$-tight graphs in which every proper subgraph is $(2,2)$-sparse. The $(2,3)$-tight variant was explored in [2].
(4) Frameworks on surfaces [9], [15]. Here it is required to consider the construction moves applied to frameworks (rather than graphs) via geometric or linear algebra arguments.

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