Financial Crises and Contagion: Dynamical Systems Approach

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The Fields Institute for Research in Mathematical Sciences
Sojourns in Nonlinear Economics
June 2012
Outline

Goal: dynamic modeling of financial crises and systemic risk

1. Single Economy: w/ R. Douady
   - Cause: breakage of stability $\rightarrow$ bifurcation
   - Effect: contagion, systemic risk $\rightarrow$ recurrence, chaos
   - Predicting a crisis: Market Instability Indicator
   - Suggested remedies

2. Multiple Economies: w/ G. Castellacci
   - Contagion from one economy to another
   - Quantitative definition of contagion
   - Suggested remedies
Single Economy Five Agent Model

C  Consumers
F  Firms
B  Banks
I  Investors
G  Government
Generalized Single Economy Five Agent Model

Agents of Economy $i$

- $C^i$ Consumers
- $F^i$ Firms
- $B^i$ Banks
- $G^i$ Government
- $I^i$ Investors restricted to Economy $i$

**Figure**: Combined flow of funds among five agents in economy $i$
Flows of Funds: Scheduled vs. At-will

- **Scheduled Cash Flows:**
  - Coupons
  - Installments, minimum credit card payments
  - Salaries
  - Contributions to pension plans
  - Taxes

- **At-will Cash Flows:** variable
  - Equity investments
  - Debt investments (loans, bonds)
  - Dividends
  - Consumption

Both are *variable* and subject to *dynamic* relations
More Flows of Funds: Contingent & International

- **Contingent Cash Flows:**
  - Quantitative Easing
  - Derivative Payoffs, e.g. CDS payouts

- **International Debt Investment:**
  - Interbank lending and investment
  - Investment in sovereign debt
  - Central banks’ lending to foreign banks

- **International Consumption and Trade:**
  - Direct consumption of foreign goods and services
  - International trade between firms
Flow of Funds for Two Economies

Figure: Flow of funds between economies $i$ and $j$
Stage 1 Contagion

Figure: Contagion from debtor $i$ to creditor $j$ inside eurozone.

- Contagion of “reduced flow of funds”
Figure: Contagion spills out of the eurozone
Early Bailout

Figure: Earlier stage of the eurozone crisis
Wealth Decomposition

\[ w_i(t) = \text{Wealth of Agent } i \text{ at time } t, \ (i = 1, \cdots, 5 \text{ for C, F, B, G, I}) \]

- **Equity / Debt split**
  - \[ w_i(t) = E_i(t) + D_i(t) \]
  - \[ E_i(t) = \text{Equity value} \]
  - \[ D_i(t) = \text{Debt value} \]

- **Liquid Asset / Invested Asset split**
  - \[ w_i(t) = L_i(t) + K_i(t) \]
  - \[ L_i(t) = \text{Liquidities: cash, cashables} \rightarrow \text{produces no income} \]
  - \[ K_i(t) = \text{Invested Assets: financial securities, property, equipment} \rightarrow \text{produces capital gain} \]
Wealth Dynamics

- **Debt:** \( D_i(t + 1) = (1 + r_i(t))D_i(t) + \tilde{\Delta}D_i(t + 1) \)
  
  - \( r_i(t) = \) average interest rate on debt of \( i \) at \( t \)
  - \( \tilde{\Delta}D_i(t) = \) new loans - capital reimbursement

- **Invested Asset:** \( K_i(t + 1) = (1 + \gamma_i(t))K_i(t) + \tilde{\Delta}K_i(t + 1) \)
  
  - \( \gamma_i(t) = \) internal growth factor (IRR)
  - \( \tilde{\Delta}K_i(t) = \) new investment - realization

- **Liquidity:** \( L_i(t + 1) = L_i(t) + \sum_{j \neq i}^n F_{ij}(t) - \sum_{k \neq i}^n F_{ki}(t) - \tilde{\Delta}K_i(t) \)
  
  - \( F_{ij}(t) = \) fund transferred from \( j \) to \( i \) at \( t \)
  - Can be seen as an “investment” with returns \( F_{ji}(s), s > t \)
  - \( F_{ii}(t) := \gamma_i(t)K_i(t) \)

- \( w_i(t + 1) = w_i(t) + \sum_{j=1}^n F_{ij}(t) - \sum_{k \neq i}^n F_{ki}(t) \)
Wealth Constraints

- Positive liquidities
  - $L_i(t) \geq 0$
  - Negative liquidities $\implies$ debt increase

- Maximum convertibility rate
  - $|\tilde{\Delta}K_i(t + 1)| \leq \kappa_i(t)K_i(t)$
  - There is a limit to converting invested assets to/from liquidities

- Borrowing capacity constraint
  - $D_i(t) \leq D_{i\text{max}}(t)$: one cannot borrow forever
  - $D_{i\text{max}}(t)$ depends on $w_i(t)$ and on market conditions
  - $(1 + r_i(t))D_i(t) > D_{i\text{max}}(t + 1) \implies$ default, bankruptcy
Assumptions on Variables

- Each $F_{ji}(t)$ produces $F_{ij}(s)$ ($s > t$) with uncertainty

- Under normal ( = non-crisis ) times,
  - $r_i(t), \gamma_i(t)$ are continuous
  - $\tilde{\Delta}K_i(t), \tilde{\Delta}D_i(t), \Delta L_i(t)$ are continuous
  - $L_i(t), K_i(t), D_i(t)$ are processes with continuous sample paths

- During a crisis, above not necessarily hold
  - Violent changes in variables can lead to a crisis
Maximizing Benefit I

- $U(x)$ is a utility function on gain $x$
  - $U : [a, b] \rightarrow \mathbb{R}, \ a < 0 < b$

**Figure:** Convex for losses, concave for gains

- $\mathbb{P}$: probability measure, $F(x) := \mathbb{P}[X \leq x]$
- Expected Utility Theory
  - $E[U(X)] = \int_{\mathbb{R}} U(x) \, dF(x)$
Maximizing Benefit II

Cumulative Prospect Theory: Subjective Utility (SU)

- Weighting function: \( W = \mathbb{1}_{[a,0)} W^- + \mathbb{1}_{(0,b]} W^+ \)
  \( \Rightarrow \) measures attitude toward risk

\[ SU[X] = \int_{\mathbb{R}} U(x) W'(F(x)) \, dF(x) \]

**Figure**: Overreact to unlikely event, magnifying fear factor
Non Linear Programming Problem

- Apply this to each $i$ for each $[t, t + 1]$
  - $U_i(x), \quad P = P_t w/ F_t(x) = P_t [X_i \leq x]$
  - $SU_{i,t}[X] =$ Subjective Utility of $U_i(x)$ at $t$
  - Net Subjective Utility (Investment) := SU (NPV of Investment)
    \[
    NSU_{i,t}(F_{ji}(t)) = SU_{i,t} \left[ \sum_{l<s_l\leq T} D(t, s_l)F_{ij}(s_l) - F_{ji}(t) \right]
    \]

- NLP: Max $z_i = \sum_{j=1}^{n} NSU_{i,t}(F_{ji}(t))$ sub. to
  - $L_i(t) \geq 0$
  - $|\tilde{\Delta}K_i(t + 1)| \leq \kappa_i(t)K_i(t)$
  - $\tilde{\Delta}D_i(t + 1) \leq D_{i\text{max}}(t + 1) - (1 + r_i(t))D_i(t)$
  - $1 \leq i \leq n, \quad t \geq 0$
Optimal Investment: Equilibrium State

- NLP with $n$ objective functions, $3n$ constraints

- $F_{ij}^*(t) = \text{the optimal solution, } 1 \leq i, j \leq n$

- Obtain Random dynamical system $f(X^*(t)) := X^*(t + 1)$ where $X_i(t) = (L_i(t), K_i(t), D_i(t)) \in \mathbb{R}^3$, $X = (X_1, X_2, \ldots, X_n)$

- Constraints produce nonlinear dynamics
  - In a crisis, constraints tend to be saturated
    $\Rightarrow$ the dynamics doesn’t depend on $U_i$
  - High leverage makes debt $\uparrow$, borrowing capacity $\downarrow$
    $\Rightarrow$ hit the constraints
  - Myopic risk estimation
    $\Rightarrow$ short-term statistics extrapolated to long-term risk
Perturbation Analysis

- From random to deterministic
  - Take non-random part $\tilde{f}$ of $f$ and rescale $X^*$ to constant dollar $X$
  - We get deterministic dynamical system $X(t + 1) = \tilde{f}(X(t))$
  - If $\tilde{f}$ becomes unstable, so does $f$

- There is an equilibrium (fixed point) $\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$
  - Diminishing marginal utility in closed economy
  - Every agent has become as rich as it can be
  - Brouwer fixed point theorem on a compact convex set

- Stable equilibrium (attracting fixed point): $\tilde{f}(\tilde{X}) = \tilde{X}$
  - Stable wealth stabilizes NLP constraints
  - Small changes in constraints preserve optimal solution
Elasticity Coefficient I

- Drop ”overline” from $\tilde{f}$: $X(t+1) = f(X(t))$
- $df$ is $3n \times 3n$: $f(X(t) + \delta X) \approx f(X) + df(X(t))\delta X$
- $\delta X_i = (\delta L_i, \delta K_i, \delta D_i)$, $\delta w_i = \delta L_i + \delta K_i$
- Derive a ”reduced” Jacobian $B$:
  - $\delta X'(t+1) = df(X(t))\delta X(t) = (\delta L'_i(t+1), \delta K'_i(t+1), \delta D'_i(t+1))$
  - $\delta L'_i(t+1) + \delta K'_i(t+1) = \delta w'(t+1) \equiv B(X(t))\delta w(t)$
  - $\delta w'(t+1) = B(X(t))\delta w(t)$
- Define Elasticity Coefficient: $a_{ij} = a_{ij}^+(t)$ or $a_{ij}^-(t)$
  - $a_{ij}^+(t) = \lim_{\Delta w_j \to 0^+} \frac{F_{ij}(w_j(t) + \Delta w_j) - F_{ij}(w_j(t))}{\Delta w_j}$
  - $a_{ij}^-(t) = \lim_{\Delta w_j \to 0^-} \frac{F_{ij}(w_j(t) + \Delta w_j) - F_{ij}(w_j(t))}{\Delta w_j}$
Different sign of $\Delta w_j(t)$ yields different reaction of $F_{ij}(t)$:

- Pre-Crisis C: failure to pay vs. no extra payment/savings
- Post-Crisis B: credit reduction vs. hoarding cash
- Post-Crisis F: layoff vs. hire freeze
**Market Instability Indicator**

- Elasticities vs. reduced Jacobian $B(X(t))$: 
  - $b_{ii} = 1 + a_{ii} - \sum_{k \neq i}^{n} a_{ki}$
  - $b_{ij} = a_{ij}$ for $i \neq j$
  - High leverage implies high elasticities

- *Market Instability Indicator*

  $$I(t) = \text{Max Eigenvalue of } B(X(t)) = \rho (B(X(t)))$$

  - This is *not* a Lyapunov exponent
  - $I(t) < 1$: perturbations of the system tend to be absorbed
  - $I(t) > 1$: small perturbations tend to increase when propagating
    $\implies$ Domino effect: possible Financial Crisis

- $I(t)$ can be empirically observed
  - Lagged correlations of historical series of flow of funds
Financial Crisis: Breakage of Stability I

NLP with reduced borrowing capacity:

- Maximize \( z_i = \sum_{j=1}^{n} \text{NSU}_{i,t} \left( F_{ji}(t) \right) \) subject to
  - \( L_i(t) \geq 0 \)
  - \( |\tilde{\Delta}K_i(t + 1)| \leq \kappa_i(t)K_i(t) \)
  - \( \tilde{\Delta}D_1(t + 1) \leq D_{1,\text{max}}(t + 1) - \mu - (1 + r_1(t))D_1(t) \)
  - \( \tilde{\Delta}D_i(t + 1) \leq D_{i,\text{max}}(t + 1) - (1 + r_i(t))D_i(t) \)
  - \( 2 \leq i \leq n, \ t \geq 0 \)

\[\Rightarrow\text{ obtain } \{f_{\mu}\}\]

- Perturb \( f \) by \( \{f_{\mu}\} \) to get new equilibrium \( \{\tilde{X}_{\mu}\} \)
  - As leverage increases so do entries of \( B_{\mu} \) \( \leftarrow \) elasticities
  - Hence eigenvalues of \( B_{\mu} \) increase
  - Even a small default at \( \tilde{X}_{\mu} \) will break the stability: \( I(t) > 1 \)
Financial Crisis: Breakage of Stability II

Figure: One dimensional illustration of stability change
Evolution of 2007-2009+ Crisis I

- **Cause:** breakage of stability $\implies$ bifurcation

- **Effect:** contagion, systemic risk $\implies$ recurrence, chaos
  - Securitization interconnected agents
  - “Default” spread along the feedback loop
  - Chaos in the financial crisis

- **Remedy:** getting out of recession $\implies$ QE etc.
  - Default set in: bailouts, loan restructuring, pay cut etc.
  - Agents minimize spending: new $f(\tilde{Y}) = \tilde{Y}$
  - $\tilde{Y}$ is a recession $\implies$ Eigenvalues of $B(\tilde{Y}) < 1$
  - Break the equilibrium by raising elasticities: QE etc.
  - **Targeted fund allocation** is necessary: no random handing out
Evolution of 2007-2009+ Crisis II

Global Recession
Deflation

Gov. action to prevent deflation (quant. easing)

Broken
Market Equilibrium

recurrence

- Government takes action to stay away from deflation (sink)
Agents of Global Economy I

$G$ is a global economy consists of $s$ subeconomies

- Economy $k$ has $n_k$ agents: $G$ has $n = \sum_{k=1}^{s} n_k$ agents
- $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))$: the global wealth vector
- For subeconomy $k$,
  - $w^k(t) = (w^k_1(t), w^k_2(t), \ldots, w^k_{n_k}(t))$: the wealth
  - $w^k_j(t)$ is the wealth of agent $j$ at $t$
  - $w_i(t) = w^k_i(t)$ if $i = N(k) + j$, $N(k) = \sum_{l=1}^{k-1} n_l$
  - $F_{N(k)+i,N(k)+j}(t) = F^k_{ij}(t)$
  - $b_{N(k)+i,N(k)+j}(t) = b^k_{ij}(t)$, $B^{(k)}(t) = \begin{pmatrix} b^k_{ij}(t) \end{pmatrix}$ is the Jacobian matrix
  - $a_{N(k)+i,N(k)+j}(t) = a^k_{ij}(t)$, $A^{(k)}(t) = \begin{pmatrix} a^k_{ij}(t) \end{pmatrix}$ is the elasticity matrix
Agents of Global Economy II

Between two economies $k$ and $l$,

- $F_{ij}^{kl}(t) = F_{N(k)+i,N(l)+j}(t)$
  - Flow of funds from agent $j$ of economy $l$ to agent $i$ of economy $k$ at time $t$

- $a_{ij}^{kl}(t) = a_{ij}^{kl+}(t)$ or $a_{ij}^{kl-}(t)$,
  - $a_{ij}^{kl+}(t) = \lim_{\Delta w_j^l \to 0^+} \frac{F_{ij}^{kl}(w_j^l(t) + \Delta w_j^l) - F_{ij}^{kl}(w_j^l(t))}{\Delta w_j^l}$
  - $a_{ij}^{kl-}(t) = \lim_{\Delta w_j^l \to 0^-} \frac{F_{ij}^{kl}(w_j^l(t) + \Delta w_j^l) - F_{ij}^{kl}(w_j^l(t))}{\Delta w_j^l}$

- Local $A^{(k)}(t)$ can be canonically embedded into the global $A(t)$
- Local $B^{(k)}(t)$ cannot be canonically embedded into the global $B(t)$
Elasticity Matrix for Multi Economy

\[ A(t) = \begin{pmatrix}
A^{(1)}(t) & A^{(12)}(t) & \ldots & A^{(1s)}(t) \\
A^{(21)}(t) & A^{(2)}(t) \\
\vdots & \vdots & \ddots & \vdots \\
A^{(s1)}(t) & \ldots & \ldots & A^{(s)}(t)
\end{pmatrix} \]

- \[ A^{(kl)}(t) = \left( a_{ij}^{kl}(t) \right)_{1 \leq i \leq n_k}^{1 \leq j \leq n_l} \]
- Global matrix is canonical embeddeddings of local matrices
Jacobian Matrix for Multi Economy

\[
B(t) = \begin{pmatrix}
\tilde{B}^{(1)}(t) & A^{(12)}(t) & \ldots & A^{(1s)}(t) \\
A^{(21)}(t) & \tilde{B}^{(2)}(t) & & \\
\vdots & & \ddots & \\
A^{(s1)}(t) & \ldots & & \tilde{B}^{(s)}(t)
\end{pmatrix}
\]

- \( \tilde{B}^{(k)}(t) \neq B^{(k)}(t) \)
- Off-diagonal block matrices \( A^{ij}(t) \) \((i \neq j)\) cause contagion
Quantitative Definition of Contagion

- We say that *contagion* in a global economic system occurs if given two times $0 < t_0 < t_1$,

1. At $t < t_0$, $\max_k \rho \left( B^{(k)}(t) \right) < 1$ and $\rho \left( B(t) \right) < 1$
2. At $t \in (t_0, t_1)$, $\max_k \rho \left( B^{(k)}(t) \right) > 1$ and $\rho \left( B(t) \right) < 1$
3. At time $t > t_1$ $B(t) \neq \bigoplus_{k=1}^{s} B^{(k)}(t)$ and $\rho \left( B(t) \right) > 1$.

- Unrelated simultaneous crises are ruled out:
  - $B(t) = \bigoplus_{k=1}^{s} B^{(k)}(t)$, then $\rho \left( B(t) \right) = \max_k \rho \left( B^{(k)}(t) \right)$
    $\implies$ independent occurrence of sub-systemic crises.
  - Condition 3 implies nonzero off-diagonal block matrices $A^{ij}(t)$ ($i \neq j$)
2010-2011+ Eurozone Crisis I

- Mini Eurozone and Mini Global Economy
  - Group I: Greece (1), Ireland (2), Portugal (3), Spain (4), and Italy (5)
  - Group II: France (6), Germany (7)
  - Group III: USA (8)

- Each economy has 5 agents: C, F, B, G, I (1 - 5)

$$A(t) = \begin{bmatrix}
A^{(1)}(t) & \cdots & A^{(16)}(t) & A^{(17)}(t) & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
A^{(61)}(t) & \cdots & A^{(67)}(t) & A^{(68)}(t) & 0 \\
A^{(71)}(t) & \cdots & A^{(76)}(t) & A^{(77)}(t) & 0 \\
0 & \cdots & A^{(86)}(t) & A^{(87)}(t) & A^{(8)}(t)
\end{bmatrix}$$

$$B(t) = \begin{bmatrix}
\tilde{B}^{(1)}(t) & \cdots & A^{(16)}(t) & A^{(17)}(t) & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\tilde{B}^{(6)}(t) & \cdots & A^{(67)}(t) & A^{(68)}(t) & 0 \\
A^{(71)}(t) & \cdots & A^{(76)}(t) & \tilde{B}^{(7)}(t) & A^{(78)}(t) \\
0 & \cdots & A^{(86)}(t) & A^{(87)}(t) & \tilde{B}^{(8)}(t)
\end{bmatrix}$$
Scenario 1. Greek sovereign debt is restructured

- Payments from Greek G to French B ↓: $F_{34}^{61} \downarrow \Rightarrow a_{34}^{61} \downarrow$
  - Entries of $A^{61}(t)$ kept low $\Rightarrow$ Little impact on $\rho(B(t))$

- Payments from Greek G to German B ↓: $F_{34}^{71} \downarrow \Rightarrow a_{34}^{71} \downarrow$
  - Entries of $A^{71}(t)$ kept low $\Rightarrow$ Little impact on $\rho(B(t))$

- Fear for French, German banks’ insolvency rises

- Markets reduce their exposure to French, German banks

- ECB & Fed’s lending to French, German banks ↑

- Post-Lehman Brothers type credit crunch is possible
Scenario 2. Greek sovereign debt is not restructured

- Domestically:
  - French, German banks more susceptible to liquidity crunches
  - $a_{i3}^{6+} \neq a_{i3}^{6-}$, $a_{i3}^{7+} \neq a_{i3}^{7-}$: hoard cash

- Externally:
  - $a_{33}^{76} \uparrow$ and $a_{33}^{86} \uparrow$: greater default risk of French banks to their German and the US counterparties
  - $a_{33}^{67} \uparrow$ and $a_{33}^{87} \uparrow$: greater default risk of German banks to their French and the US counterparties
  - These belong to the off-diagonal blocks $B(t)$
  - Higher probability for $\rho(B(t)) > 1 \implies$ Global financial crisis
Scenario 3. Fear Factor

- If French B and I lose confidence in Italian sovereign debt:
  - $\text{NSU}_{3,t}^6 \left( F_{43}^{56} (t) \right)$ decreases $\implies F_{43}^{56}$ decreases
  - $\text{NSU}_{5,t}^6 \left( F_{45}^{56} (t) \right)$ decreases $\implies F_{45}^{56}$ decreases

- If German B and I lose confidence in Italian sovereign debt:
  - $\text{NSU}_{3,t}^7 \left( F_{43}^{57} (t) \right)$ decreases $\implies F_{43}^{57}$ decreases
  - $\text{NSU}_{5,t}^7 \left( F_{45}^{57} (t) \right)$ decreases $\implies F_{45}^{57}$ decreases

- Italian sovereign bond yields soar
- Risk of Italian default rises
- This is not due to *Contagion*
Current Issues

- Scenario 1 does not work well
  - Greek G (now CCC) cannot print euro
  - Austerity deepens recession

- What if Greece is to leave eurozone?
  - Worst: all three scenarios for all major economies
    \[ \Rightarrow \text{situation grows exponentially worse} \]
  - Hope: depending on exit strategy, things may improve
  - Greek debt: terms of restructuring, then currency control
  - The rest: keep money flow (≠ printing more)

- Group 2: French politics

- Group 3: U.S. banks
1997-98 Asian-Russian Crisis

Figure: Flow of funds vs. flow of default among stricken countries and foreign investors

- Each country could devalue its own currency
- Off-block matrices $A^{kl}(t)$ are zero $\implies$ no contagion
Conclusion: Work in Progress

- **Theoretical**
  - Analyze the crisis dynamics
  - Impact of hitting borrowing and liquidity constraints
  - Impact of Government actions: Quantitative easing, taxes, expenditures, bail out, etc.

- **Empirical**
  - Collect and sort out Flow of Funds data
  - Simulate Instability Indicator
  - Validate the hypothesis that it anticipates systemic crises
  - Simulate Government actions
Selected References


