The geometry of the groups $\mathrm{PSL}(2, q)$

Julie De Saedeleer

Université Libre de Bruxelles

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Classification of the finite simple groups

- **Context**: Classification of the finite simple groups.
  - 18 infinite families.
  - 26 sporadic groups.
- **Question left open**: To achieve a unified geometric interpretation of all finite simple groups (Buekenhout).
- **Encouragement in this direction**:
  Theory of Buildings by J. Tits.
  Applies to 17 of the 18 infinite families
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1. Classify geometries over a given diagram
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Sample of known results

Sample of known results, theorical and experimental

- Every $Alt(n)$ and $Sym(n)$ for $n \leq 8$ (Cara)
- Sporadic groups (Buekenhout, Dehon, Gottchalk, Leemans, Miller):
  $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL$
  O'Nan (partial results)
- Sz (Leemans)
- Every $\mathrm{PSL}(2, q)$ for $q \leq 19$ (Cara, Dehon, Leemans, Vanmeerbeek)

On the way to classify all geometries of $\mathrm{PSL}(2, q)$

Idea

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Classification of all coset geometries of **rank two** on which some group $\mathrm{PSL}(2, q)$, q a prime power, acts flag-transitively.

Classification under additional conditions, to be explained.

Incidence geometry of rank two

Geometry of rank two

A *geometry* Γ is a four-tuple $(X, *, t, I)$ where

1. X is a set whose elements are called the *elements of* Γ*;*
2. *I is the set $\{0, 1\}$ whose elements are called the types of Γ;*
3. *$t : X \to I$ is a mapping from X onto I;*
4. *$*$ is a symmetric and reflexive relation on $X \times X$ such that no two distinct elements of the same type are incident.*

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Flag

In a geometry, a *flag* $\mathcal{F}$ is a set of pairwise incident elements.

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Construction of a Coset geometry for $(G, \{G_0, G_1\})$

We construct a geometry $\Gamma = \Gamma(G, (G_i)_{i \in I}) = (X, t, *, I)$ as follows

1. The set of elements is $X = \{gG_i | g \in G, G_i \in (G_i)_{i \in I}\}$.
2. We define an *incidence relation* $*$ on $X \times X$ by

$$gG_i * hG_j \Leftrightarrow gG_i \bigcap hG_j \neq \emptyset$$

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Set of axioms on the geometries

Let $\Gamma = \Gamma(G; \{G_0, G_1\})$ be a geometry of rank two:

- the geometry Γ must be *firm (F)*;
- the geometry Γ must be *residually connected (RC)*;
- the group G must act *flag-transitively (FT)* on Γ;
- the group G must act *residually weakly primitively (RWPRI)* on Γ;
- the geometry Γ must be *locally two-transitive* $(2T)_1$.

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Lemmas

1. If Γ is a rank two geometry, then G acts FT on Γ.
2. If Γ is RWPRI, then it is also firm and RC.

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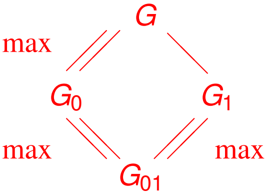
The only axioms we must verify for the rank two are: RWPRI and $(2T)_1$.

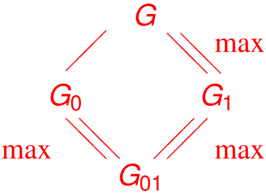
RWPRI and $2T_1$

Let $\Gamma(G; \{G_0, G_1\})$ be a geometry of rank 2.

- RWPRI



or



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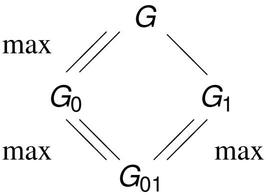

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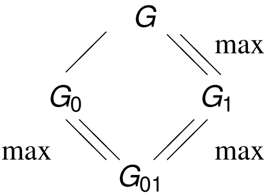

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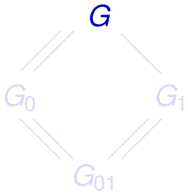
Method

1. Enumerate the possible configurations



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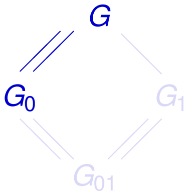

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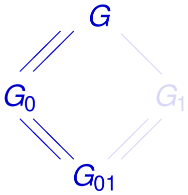

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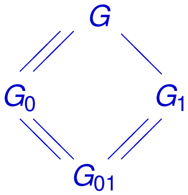

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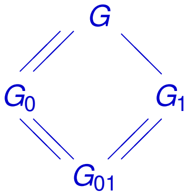

2. Count the geometries:
   - up to isomorphism
   - up to conjugacy

Classification Theorem of rank two geometries of $\mathrm{PSL}(2, q)$ groups

Classification Theorem (DS and Leemans)

Let $G \cong \mathrm{PSL}(2, q)$ and $\Gamma(G; \{G_0, G_1, G_0 \cap G_1\})$ be a locally two-transitive RWPRI geometry of rank two. If G_0 is isomorphic to one of $E_q : \frac{(q-1)}{(2,q-1)}$, $D_{2\frac{(q\pm1)}{(2,q-1)}}$, A_4, S_4, A_5, $\mathrm{PSL}(2, q_i)$ or $\mathrm{PGL}(2, q_i)$, then Γ is isomorphic to one of the geometries appearing in the following tables

	$G_0 \cong S_4$			$q = p > 2$ and $q = \pm 1(8)$
G_{01}	G_1	$\sharp$ up to conj.	$\sharp$ up to isom.	Extra conditions on q
D_6	D_{12}	2	1	$q = \pm 1(24)$
	D_{18}	2	1	$q = \pm 1(72)$ or $q = \pm 17(72)$
	S_4	2	1	$\frac{q \pm 1}{6}$ even
	S_4	1	1	$\frac{q \pm 1}{6}$ odd
D_8	D_{16}	2	1	$q = \pm 1(16)$
	D_{24}	2	1	$q = \pm 1(24)$
	S_4	2	1	$\frac{q \pm 1}{8}$ even
	S_4	1	1	$\frac{q \pm 1}{8}$ odd
A_4	A_5	2	1	$q = \pm 1(40)$ or $q = \pm 9(40)$

Table: The *RWPRI* and $(2T)_1$ geometries with $G_0 = S_4$.

Conclusion

- The classification of the rank two geometries under the given conditions is complete.
- Our list comprises infinite classes of geometries up to conjugacy (resp. isomorphism) depending on the prime power p^n.
- If $q \leq 97$ there are 329 geometries up to conjugacy and 190 geometries up to isomorphism.

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Perspectives

Idea

Classify all incidence geometries for $\mathrm{PSL}(2, q)$ groups under the given axioms.

Steps

1. Classify all geometries of **rank two.**
2. What is the maximal rank?
3. Classify all geometries with no restriction on the rank.
4. Use another axiom to reduce the number of geometries:

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Self-normalizing or a slightly weaker version
Borel self-normalizing

Self-normalizing and Borel-self-normalizing

Conclusions on BSN-SN

- Under BSN, there exists no geometry in which G_0 is isomorphic to A_4, $E_q : \frac{q-1}{(2,q-1)}$.
- Under SN, there exists no geometry in which G_0 is isomorphic to A_4, $E_q : \frac{q-1}{(2,q-1)}$ and $\mathrm{PSL}(2,q)$ over a subfield.
- For $q \leq 97$, BSN (resp. SN) leaves only 42 (resp. 36) geometries out of 190, up to isomorphism.
- If we impose BSN on higher ranks we restrict the number of possible maximal parabolic subgroups to 3, namely: $E_q : \frac{q-1}{(2,q-1)}$, $\mathrm{PSL}(2,q')$ and $\mathrm{PGL}(2,q')$ (over a subfield).
- $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS$ and McL have at least one geometry satisfying SN.
- If we apply SN to the classification of RWPRI geometries for the Sz groups only the Building remains.

Self-normalizing and Borel-self-normalizing

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Locally s-arc-transitive graphs: Context

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Idea:

Given a group $G \cong \mathrm{PSL}(2, q)$,
use the classification of RWPRI and $(2T)_1$ rank 2 geometries to obtain for each geometry the highest value of s such that the incidence graph is locally s-arc-transitive.

Sample of earlier work

Sample of earlier work:

- Sz (Leemans, 1998 and Praeger-Fang, 1999)
- Ree (Praeger, Fang and Li, 2004)
- Sporadic groups (Leemans, 2009):
  $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, He, Ru, Suz, Co_3$
  $O'Nan$ (partial results)

locally s-arc-transitive graph

Let $\mathcal{G}(V, E)$ be a finite simple undirected connected graph.

- An s-arc is an $(s+1)$-tuple $(\alpha_0, ..., \alpha_s)$ of vertices such that $\{\alpha_{i-1}, \alpha_i\}$ is an edge of $\mathcal{G}$ for all $i = 1, ..., s$ and $\alpha_{j-1} \neq \alpha_{j+1}$ for all $j = 1, ..., s-1$.
- Given $G \leq \mathrm{A}ut(\mathcal{G})$.
  We call $\mathcal{G}$ locally (G, s)-arc-transitive if $\mathcal{G}$ contains an s-arc and given any two s-arcs α and β starting at the same vertex v, there exists an element $g \in G_v$ mapping α to β.

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- An s-arc is an $(s+1)$-tuple $(\alpha_0, ..., \alpha_s)$ of vertices such that $\{\alpha_{i-1}, \alpha_i\}$ is an edge of $\mathcal{G}$ for all $i = 1, ..., s$ and $\alpha_{j-1} \neq \alpha_{j+1}$ for all $j = 1, ..., s-1$.
- Given $G \leq \mathrm{A}ut(\mathcal{G})$.
  We call $\mathcal{G}$ locally (G, s)-arc-transitive if $\mathcal{G}$ contains an s-arc and given any two s-arcs α and β starting at the same vertex v, there exists an element $g \in G_v$ mapping α to β.

Giudici, Li, Praeger, 2004

Search for graphs $\mathcal{G}$ having G acting as a locally 2-arc-transitive automorphism group is equivalent to determining the pairs of subgroups $\{G_0, G_1\}$ in G such that

- (P_1) G_0 (resp. G_1) has a 2-transitive action on the cosets of $B = G_0 \cap G_1$ in G_0 (resp. G_1)
  (this ensures local 2-arc-transitivity) $\Leftrightarrow$ $(2T)_1$;
- (P_2) $\langle G_0, G_1 \rangle = G$
  (this ensures connectedness of the graph) $\Leftrightarrow$ RC ;
- (P_3) $B = G_0 \cap G_1$ is core-free in G.
  This is clearly satisfied since $G \cong \mathrm{PSL}(2, q)$ is simple.

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The Algorithm of Tits shows that:

These locally $(G, 2)$-arc-transitive graphs are rank two geometries.

All rank two geometries satisfying $(2T)_1$ and *RWPRI* classified in the Theorem satisfy (P_1),(P_2) and (P_3)

$$\Downarrow$$

They are locally 2-arc-transitive graphs.

Aim

For every geometry Γ given in the classification Theorem, we try to determine the highest value of s such that the incidence graph of Γ is a locally s-arc-transitive graph.

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For every geometry Γ given in the classification Theorem, we try to determine the highest value of s such that the incidence graph of Γ is a locally s-arc-transitive graph.

Conclusion

Most values of s are 2 or 3.

Open problem

- In a few cases, we only get a set of possible values for s.
- The exact value may be computed by Magma but only for small values of q.

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Examples:

- $\Gamma(\mathrm{PSL}(2,q); S_4, S_4, D_8)$ $s = 4$ for the values $q = 9, 17, 23, 31, 41, 47, 71, 73, 79, 89$;
- $\Gamma(\mathrm{PSL}(2,q); S_4, D_{16}, D_8)$ $s = 7$ for the values $q = 17, 31, 79, 97$.

The end

Thank you!

Incidence graph

For each pair of subgroups $\{G_0, G_1\}$ of G satisfying (P_1), (P_2) and (P_3)

- the corresponding graph is the incidence graph $\mathcal{G}$ of Γ which is the graph whose vertices are the left cosets of the subgroups $(G_i)_{i \in I}$.
- Two vertices are joined provided the corresponding cosets have a non-empty intersection.
- The type of a vertex $v = gG_i$ of the incidence graph is i.

Lemma (Leemans 2009)

Let G be a group and $\{G_0, G_1\}$ be a pair of subgroups satisfying properties (P_1), (P_2) and (P_3). Denote by b_i the index of $B := G_0 \cap G_1$ in G_i (with $i = 0, 1$). If $(G; G_0, G_1, G_{01})$ is a locally s-arc-transitive graph (with $s \geq 2$), then

- $((b_0 - 1)(b_1 - 1))^{\frac{s-1}{2}}$ divides $|B|$ if s is odd;
- $((b_0 - 1)(b_1 - 1))^{\frac{s-2}{2}}.lcm(b_0 - 1, b_1 - 1)$ divides $|B|$ if s is even,

where $lcm(b_0 - 1, b_1 - 1)$ is the lowest common multiple of $b_0 - 1$ and $b_1 - 1$.

Corollary (Leemans 2009)

If $(G; G_0, G_1, G_{01})$ is a locally s-arc-transitive graph with $B := G_0 \cap G_1$ a cyclic group of prime order and with at least one b_i not equal to 2, then s is at most 3. Moreover, if $s = 3$, then one of b_0 or b_1 must be equal to 2.

Observe also that the SN property implies the BSN property. In the classification of RWPRI and $(2T)_1$ geometries of rank two, the only geometries that satisfy the BSN property but do not satisfy the SN property are

$$\Gamma\left(\mathrm{PSL}(2,2^{2n});\mathrm{PSL}(2,2^n),E_{2^{2n}}:(2^n-1),E_{2^n}:(2^n-1)\right)\text{ with }n\neq 1$$

and

$$\Gamma\left(\mathrm{PSL}(2,p^{2n});\mathrm{PGL}(2,p^n);E_{p^{2n}}:(p^n-1);E_{p^n}:(p^n-1)\right)$$

$$\text{with } p \text{ odd prime .}$$

	$G_0 \cong S_4$			$q = p > 2$ and $q = \pm 1(8)$
G_{01}	G_1	BSN	SN	Extra conditions on q
D_6	D_{12}	no	no	$q = \pm 1(24)$
D_6	D_{18}	no	no	$(q = \pm 1(72)$ or $q = \pm 17(72))$ and $\frac{q\pm 1}{18}$ even
D_6	D_{18}	yes	yes	$(q = \pm 1(72)$ or $q = \pm 17(72))$ and $\frac{q\pm 1}{18}$ odd
D_6	S_4	no	no	$\frac{q\pm 1}{6}$ even
D_6	S_4	yes	yes	$\frac{q\pm 1}{6}$ odd
D_8	D_{16}	no	no	$q = \pm 1(16)$
D_8	D_{24}	no	no	$q = \pm 1(24)$ even
D_8	D_{24}	yes	yes	$q = \pm 1(24)$ odd
D_8	S_4	no	no	$\frac{q\pm 1}{8}$ even
D_8	S_4	yes	yes	$\frac{q\pm 1}{8}$ odd
A_4	A_5	no	no	$q = \pm 1(40)$ or $q = \pm 9(40)$

Table: The RWPRI and $(2T)_1$ geometries with $G_0 \cong S_4$.

	$G_0 \cong S_4$		$q = p > 2$ and $q \equiv \pm 1(8)$
G_{01}	G_1	locally(G, s)-arc-transitive graphs	Extra conditions on q
D_6	D_{12}	$s = 3$	$q \equiv \pm 1(24)$
D_6	D_{18}	$s = 2 or 3$	$q \equiv \pm 1(72)$ or $q \equiv \pm 17(72)$
D_6	S_4	$s = 2$	$q \equiv \pm 1(6)$
D_8	D_{16}	$s = 3, 5 or 7$	$q = \pm 1(16)$
D_8	D_{24}	$s = 2, 3 or 4$	$q \equiv \pm 1(24)$
D_8	S_4	$s = 2, 3 or 4$	none
A_4	A_5	$s = 3$	$q \equiv \pm 1(40)$ or $q \equiv \pm 9(40)$

Table: locally s-arc-transitive graphs that are not locally $(s+1)$-arc-transitive with $G_0 \cong S_4$.