Computing Cohomology Ring

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Outline

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Background from *Cubical Homology*

A *cubical set* $X \subset \mathbb{R}^d$ is a finite union of *elementary cubes*

$$Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d,$$

I_i of the form $[k, k+1]$ or $\{k\}$.
$\mathcal{K}(\mathbb{R}^d)$ all such Q
$\mathcal{K}_k(\mathbb{R}^d)$ those of dimension k
$\mathcal{K}_k(X)$ those in X
$C_k(X)$ free abelian group generated by
$\mathcal{K}_k(X)$, its canonical basis

$$\partial_k(Q) = \sum \pm(k-1)\text{–dim faces of } Q$$

cubical boundary operator
Alternation defined in *[CH]* by induction on $d = \operatorname{emb} X$

Cubical cochain complex

$\mathcal{C}(X)$ *cubical chain complex*

Note: All is valid for $R(\mathcal{K}(X)) := \mathcal{C}(X; R)$ graded module with coefficients in a ring with unity R.

$(\mathcal{C}^*(X), \delta)$ the *dual complex* of $(\mathcal{C}_*(X), \partial)$

$\delta^k : C^k(X) \to C^{k+1}(X)$ the *coboundary operator*

$$\langle \delta^k c^k, c_{k+1} \rangle := \langle c^k, \partial_{k+1} c_{k+1} \rangle$$

$\mathcal{K}^k(X) := \{Q^\star | Q \in \mathcal{K}_k(X)\}$ the dual canonical basis of $C^k(\mathbb{R}^d)$

$$H^k(X) := Z^k(X)/B^k(X) = \ker \delta^k / \mathrm{im}\, \delta^{k-1}$$

the k'th *cohomology group* of X

Cubical cross product

Recall from *CH*: the *cubical product*

$$\times : C_p(\mathbb{R}^n) \times C_q(\mathbb{R}^m) \to C_{p+q}(\mathbb{R}^{n+m})$$

is defined on $P \in \mathcal{K}_p^n$ and $Q \in \mathcal{K}_q^m$ as the cartesian product $P \times Q$ and extended to chains (c, c') by bilinearity.

The *cubical cross product* of cochains
$c^p \in C^p(X)$ and $c^q \in C^q(Y)$
is a cochain in $C^{p+q}(X \times Y)$
defined on $R \times S \in \mathcal{K}_{p+q}(X \times Y)$ by

$$\langle c^p \times c^q, Q \rangle := \begin{cases} \langle c^p, R \rangle \cdot \langle c^q, S \rangle & \text{if } \dim R = p \text{ and } \dim S = q, \\ 0 & \text{otherwise.} \end{cases}$$

Cup product – general definition

The *cubical cup product*

$$\smile : C^p(X) \times C^q(X) \to C^{p+q}(X)$$

of cochains c^p and c^q is defined on $Q \in \mathcal{K}_{p+q}(X)$ by:

$$\langle (c^p \smile c^q), Q \rangle := \langle \mathrm{diag}^{p+q}(c^p \times c^q), Q \rangle = \langle c^p \times c^q, \mathrm{diag}_{p+q}(Q) \rangle$$

where diag_{p+q} is the *homology chain map* induced by $\mathrm{diag}(x) = (x, x)$.

The *cup product* $\smile : H^p(X) \times H^q(X) \to H^{p+q}(X)$ is defined by

$$[z^p] \smile [z^q] := [z^p \smile z^q].$$

Our goal: Obtain an explicit formula suitable for computations.

Chain map of the diagonal map

Theorem (Chain selector CH)

Given $f : X \to Y$*, let* $F : X \rightrightarrows Y$*,*

$$F(x) := \mathrm{ch}\,(f(\mathrm{ch}\,(x)),$$

where ch(A) *is the smallest cubical set containing* A*.*
Suppose F *acyclic–valued. Then* $\exists$ *a chain map* $\varphi : \mathcal{C}(X) \to \mathcal{C}(Y)$ *such that*

(a) $|\varphi_k(Q)| \subset (F(\overset{\circ}{Q}))\ \forall_{Q \in \mathcal{K}_k(X)}$*;*

(b) $\varphi_0(V) \in \mathcal{K}_0\ \forall_{Q \in \mathcal{K}_0(X)}$*.*

For any chain map satisfying (a,b), we have

$$\varphi_* = H_*(f).$$

Lemma (1)

Let X_1, X_2, Y_1, Y_2 be cubical sets,
$f : X_1 \to Y_1$, $g : X_2 \to Y_2$ maps which admit acyclic-valued representations F, G, φ and ψ their chain selectors.

1. *The set-valued map $F \times G : X_1 \times X_2 \rightrightarrows Y_1 \times Y_2$ given by*

$$(F \times G)(x, y) := F(x) \times G(y)$$

   is an acyclic-valued representation of $f \times g$.

2. *The chain map $\varphi \otimes \psi : \mathcal{C}(X_1 \times X_2) \to \mathcal{C}(Y_1 \times Y_2)$ given on generators $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X_1 \times X_2)$ by*

$$(\varphi \otimes \psi)_k(Q) := \varphi_p(Q_1) \times \psi_q(Q_2)$$

   is a chain selector of $F \times G$.

Theorem (2)

Let X, Y be cubical sets and let $\lambda : X \times Y \to Y \times X$ be the transpose given by $\lambda(x, y) := (y, x)$.

1. *The map $\Lambda : X \times Y \rightrightarrows Y \times X$ given by*

$$\Lambda(x, y) := Q_2 \times Q_1, \;\; Q_1 := \mathrm{ch}\,(x), Q_2 := \mathrm{ch}\,(y),$$

   is an acyclic-valued representation of λ;

2. *Let $\lambda_\# : \mathcal{C}(X \times Y) \to C(Y \times X)$ be defined on generators $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X \times Y)$ by*

$$\lambda_k(Q) := (-1)^{\dim Q_1 \dim Q_2} Q_2 \times Q_1 .$$

   Then $\lambda_\#$ is a chain selector of Λ.

Corollary (3)

Let $\tau : (X_1 \times Y_1) \times (X_2 \times Y_2) \to (X_1 \times X_2) \times (Y_1 \times Y_2)$ *be given by* $\tau(x_1, y_1, x_2, y_2) = (x_1, x_2, y_1, y_2)$. *Then*

1. $T(x) := \tau(\mathrm{ch}\,(x))$ *is an acyclic-valued representation of* τ*;*
2. *The map*

$$\tau_\# : \mathcal{C}(X_1 \times Y_1 \times X_2 \times Y_2) \to \mathcal{C}(X_1 \times X_2 \times Y_1 \times Y_2)$$

defined on generators by

$$\tau_\#\,((P_1 \times P_2) \times (Q_1 \times Q_2)) := (-1)^{\dim P_2 \dim Q_1}\,(P_1 \times Q_1) \times (P_2 \times Q_2)$$

is its chain selector.

Constructing $\varphi = \text{diag}_\#$

Proposition (4)

The map diag : $X \to X \times X$ *admits an acyclic-valued representation* Diag : $X \rightrightarrows (X \times X)$ *given by* $\text{Diag}(x) := Q \times Q$ *where* $Q = \mathrm{ch}\,(x) \in \mathcal{K}(X)$.

We go by induction on $d = \mathrm{emb}\,(X)$.

Case $d = 1$:

$$Q = [v] \;\Rightarrow\; \text{diag}_0([v]) := [v] \times [v]$$

$$Q = [v_0, v_1] \;\Rightarrow\; \text{diag}_1([v_0, v_1]) := [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1]$$

Induction step:

$$\begin{aligned}\operatorname{diag}(x_1,\dots,x_d) &= (x_1,\dots,x_d,x_1,x_2,\dots,x_d)\\ &= \tau(x_1,x_1,x_2,\dots,x_d,x_2,\dots,x_d)\\ &= \tau(\operatorname{diag}(x_1),\operatorname{diag}(x_2,\dots,x_d))\end{aligned}$$

We get

$$\operatorname{diag} = \tau \circ (\operatorname{diag}^{X_1} \times \operatorname{diag}^{X'}) \circ j,$$

where τ is the permutation,

$$j : X \hookrightarrow X_1 \times X_2 \subset \mathbb{R} \times \mathbb{R}^{d-1}$$

the inclusion of X to the product of its projections.

Theorem (5)

Let $\mathrm{emb}\ X > 1$*. Define* $\mathrm{diag}_\# : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ *by*

$$\mathrm{diag}_\# := \pi \circ \tau_\# \circ (\mathrm{diag}_\#^{X_1} \otimes \mathrm{diag}_\#^{X'}) \circ \iota,$$

where

1. $\iota : \mathcal{C}(X) \to \mathcal{C}(X_1 \times X')$ *is the inclusion map;*
2. $\mathrm{diag}_\#^{X_1}$ *and* $\mathrm{diag}_\#^{X'}$ *are defined by induction hypothesis;*
3. $\mathrm{diag}_\#^{X_1} \otimes \mathrm{diag}_\#^{X'}$ *is give in Lemma 1;*
4. $\tau_\#$ *is given in Corollary 3;*
5. $\pi : \mathcal{C}(X_1 \times X' \times X_1 \times X') \to \mathcal{C}(X \times X)$ *is the projection*

$$\pi(Q) := \begin{cases} Q & \text{if } Q \in \mathcal{K}(X \times X), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{diag}_\#$ *is a chain selector for* Diag.

Explicit cup product formula

Induction on $d = \mathrm{emb}\,(X)$. Case $d = 1$:

$k = p + q = 0$, $R = [v]$. Then

$$\begin{aligned} \langle P^\star \smile Q^\star, R\rangle &= \langle P^\star \times Q^\star, [v]\times[v]\rangle \\ &= \begin{cases} 1 & \text{if } P = Q = [v], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$k = p + q = 1$, $R = [v_0, v_1]$. Then

$$\begin{aligned} \langle P^\star \smile Q^\star, R\rangle &= \langle P^\star \times Q^\star, [v_0]\times[v_0, v_1] + [v_0, v_1]\times[v_1]\rangle \\ &= \begin{cases} 1 & \text{if } P = [v_0] \text{ and } Q = [v_0, v_1], \\ 1 & \text{if } P = [v_0, v_1] \text{ and } Q = [v_1], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem (6)

Let $X \subset \mathbb{R}$ and let $P, Q \in \mathcal{K}(X)$, $P = [a, b]$, $Q = [c, d]$ be elementary intervals, possibly degenerated. Then

$$P^\star \smile Q^\star = \begin{cases} [a]^\star & \text{if } a = b = c = d, \\ [c, d]^\star & \text{if } a = b = c = d - 1, \\ [a, b]^\star & \text{if } b = c = d = a + 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $P^\star \smile Q^\star$ is either zero or a dual of an elementary interval.

Example Let $X = [a, a+1] \in \mathbb{R}$. We have

$[a]^\star \smile [a, a+1]^\star = [a, a+1]^\star$ and $[a, a+1]^\star \smile [a]^\star = 0$.

Hence the graded commutative law for cohomology classes does not hold for chain complexes.

Induction step

Theorem (7)

Let $\mathrm{emb}\, X = d > 1$*, and suppose that the formula for* $\smile$ *is given for cochains with* $\mathrm{emb}\ = 1, \ldots, d-1$*.*
Let $P = P_1 \times P_2 \in \mathcal{K}_p(X)$ *and* $Q = Q_1 \times Q_2 \in \mathcal{K}_q(X)$ *with* $\mathrm{emb}\ P_1 = \mathrm{emb}\ Q_1 = 1$ *and* $\mathrm{emb}\ P_2 = \mathrm{emb}\ Q_2 = d-1$*.*
Let $x = P_1^\star \smile Q_1^\star$*,* $y = P_2^\star \smile Q_2^\star$ *be computed using induction. Then*

$$P^\star \smile Q^\star = \begin{cases} (-1)^{\dim P_2 \dim Q_1}\ x \times y & \text{if } |x \times y| \in \mathcal{K}(X), \\ 0 & \text{otherwise.} \end{cases}$$

Example Let $X = [0,1]^2$, $P = [0] \times [0,1]$, and $Q = [0,1] \times [0]$.

$$P^\star \smile Q^\star = (-1)^{1 \cdot 1}\left([0]^\star \smile [0,1]^\star\right) \times \left([0,1]^\star \smile [1]^\star\right) = -[0,1]^{2\star}.$$

However, if $X = P \cup Q$, we get $P^\star \smile Q^\star = 0$.

Coordinate-wise formula

Let

$$P = I_1 \times I_2 \times \cdots \times I_d \text{ and } Q = J_1 \times J_2 \times \cdots \times J_d.$$

Let $P'_j := I_{j+1} \times I_{j+2} \times \cdots \times I_d$. Put

$$\operatorname{sgn}(P, Q) := (-1)^{\sum_{j=1}^{d} \dim P'_j \dim J_j} = (-1)^{\sum_{j=1}^{d} \left(\dim J_j \sum_{i=j+1}^{d} \dim I_i\right)}.$$

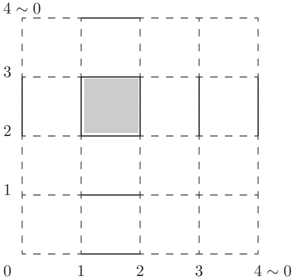
Corollary (8)

With the above notation,

$$P^\star \smile Q^\star = \operatorname{sgn}(P, Q)(I_1^\star \smile J_1^\star) \times (I_2^\star \smile J_2^\star) \times \cdots \times (I_d^\star \smile J_d^\star),$$

provided the right-hand side is supported in X, and $P^\star \smile Q^\star = 0$ otherwise.

Example – cubical torus



$T = \left(\mathrm{bd}\,[0,1]^2\right)^2 \cong ([0,4]/\sim)^2$

$x^1 = \sum(\text{solid line vertical edges})^\star, \quad x^1 = \sum(\text{horizontal ones})^\star$

$x_1^1 \smile x_2^1 = -([1,2] \times [2,3])^*$ generator of $H^2(T)$.
Its support $|x^1 \smile y^1|$ is the gray square.

Cohomology of S-complexes

S-complex defined by Mrozek *et. al.* (S, κ)
is a combinatorial setting for a CW-complex
with a chosen canonical basis $S = \{S_k\}$
and given *incidence numbers* κ in a ring of coefficients R
such that
$\kappa(a, b) \neq 0$, $a \in S_k$, then b is a $(k-1)$–face of a.
Particular cases: Any simplicial complex S,
$S = \mathcal{K}(X)$, X a cubical set.
This gives rise to the chain complex $R(S)$, its *dual*
$R^\star(S) := \mathrm{Hom}\,(R(S), R)$, the *coboundary map* $\delta^\kappa := (\partial^\kappa)^\star$,

$$\delta^\kappa(t^\star) := \sum_{s \in S} \kappa(s, t) s^\star \quad t \in S.$$

and to the *cohomology* $H^*(S)$.

Reduction and coreduction pairs

Smith normal form algorithm is costly! Our goal: We want a low-cost method for removing as many generators as possible before applying it.

An S-subcomplex S' is *closed* if $bd_S S' \subset S'$ and it is *open* if $S \setminus S'$ is closed.

Consider a pair $(a, b) \in S \times S$ with invertible $\kappa(Q, P)$.

It is a *reduction pair* if $cbd_S(b) = \{a\}$.

It is a *coreduction pair* if $bd_S(a) = \{b\}$.

Theorem (9)

A reduction pair (a, b) is open in S and a coreduction pair is closed in S. In both cases $\{a, b\}$ and $\bar{S} := S \setminus \{a, b\}$ are S-subcomplexes of S, and $H(\{a, b\}) = H^*(\{a, b\}) = 0$. Consequently,

$$H(S) \cong H(\bar{S}) \ \text{ and } \ H^*(S) \cong H^*(\bar{S}).$$

Attention: To compute the cohomology ring of a cubical set it is not sufficient to have the cohomology generators in a reduced S-complex.
It is necessary to construct the cohomology generators in the original cubical set.

Theorem (10)

Let (a, b) be a reduction or coredution pair in S. The maps $\psi = \psi^{(a,b)} : R(S) \to R(\bar{S})$, $\iota = \iota^{(a,b)} : R(\bar{S}) \to R(S)$,

$$\psi(c) = c - \frac{\langle c, a \rangle}{\langle \partial b, a \rangle} \partial b - \langle c, b \rangle b, \tag{1}$$

$$\iota(c) = c - \frac{\langle \partial c, a \rangle}{\langle \partial b, a \rangle} b \tag{2}$$

are mutually inverse chain equivalences.

Reduction sequence

is a finite sequence

$$\omega = \{(a_i, b_i)_{i=1,2,\ldots n}\} \in S$$

such that (a_i, b_i) is a reduction or coreduction pair in (S^{i-1}, κ^{i-1}),
starting at $S^0 = S$ and ending at $S^\omega := S^n$.
A *homology model of S* is S^ω together with the chain equivalences:

$$\begin{aligned}
\iota^\omega &= \iota^{(a_1,b_1)} \circ \cdots \circ \iota^{(a_n,b_n)} : R(S^\omega) \to R(S), \\
\psi^\omega &= \psi^{(a_n,b_n)} \circ \cdots \circ \psi^{(a_1,b_1)} : R(S) \to R(S^\omega).
\end{aligned}$$

Dual chain equivalences

Idea: Use $(\iota^\omega)^\star$ to transport cochains on S to those in S^ω and $(\psi^\omega)^\star$ the reverse way.

Proposition (11)

The duals of ψ and ι are given by

$$\psi^\star : R^\star(\bar{S}) \ni c \mapsto c - \frac{\langle b^\star, \delta c\rangle}{\langle \partial b, a\rangle} a^\star \in R^\star(S),$$

$$\iota^\star : R^\star(S) \ni c \mapsto c - \frac{\langle b^\star, c\rangle}{\langle \partial b, a\rangle} \delta a^\star \in R^\star(\bar{S}).$$

Using models for cohomology

Note: There are even more benefits from the homology model for cohomology computation than for homology computation!

Theorem (12)

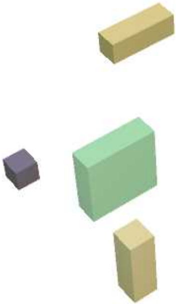
Let ω be a sequence consisting only of elementary coreduction pairs. Then $(\psi^\omega)^\star$ is an inclusion

$$R^\star(S^\omega) \hookrightarrow R^\star(S),$$

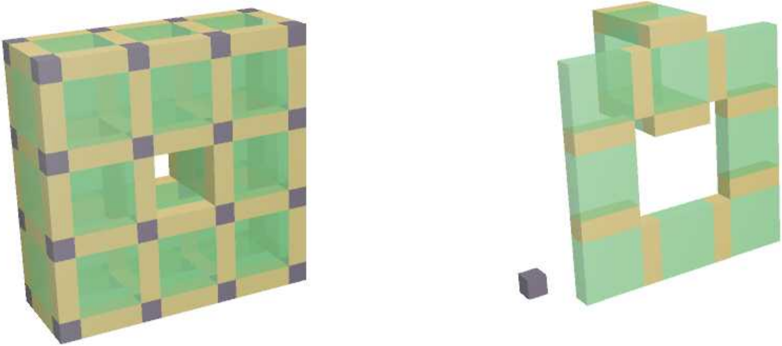
Consequence: For computing the ring structure of a cubical set X, when $S = \mathcal{K}(X)$, the cup product formula may be applied directly to the cohomology generators in the ω-reduction.

Wedge X and torus T

$X = S^2 \vee S^1 \vee S^1$. $H(X) \cong H(T)$, but $H^*(X) \ncong H^*(T)$ as rings.

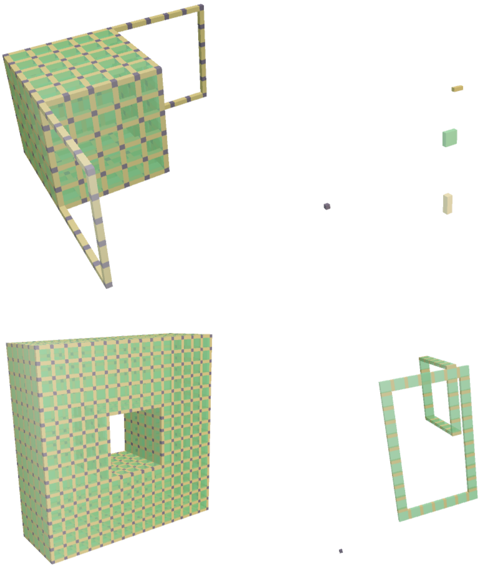


Cubical model for X. Left: original, Right: coreduced.
Cup product of H^1–generators computed easily in the coreduced complex is zero.



Cubical model for T. Left: original, Right: coreduced. Cup product of H^1–generators computed easily in the coreduced complex is a H^2–generator.

Efficiency test on rescaled X and T



Past and future work

Origins of (singular) cubical approach:
J.P. Serre, *Homologie singulière des espaces fibrés*, Annals Math. (1951).
More recent computation oriented work:
R. González-Diáz and P. Real, *Computation of cohomology operations on finite simplicial complexes*, Homology, Homotopy & Appl. (2003).
T. Kaczynski, K. Mischaikow, and M. Mrozek, *Computational Homology*, Springer 2004.
M. Mrozek and B. Batko, *Coreduction homology algorithm*, Discrete & Comput. Geom. (2009).
Related program libraries *CHomP, CAPD-RedHom*.
Future work:
Implementation and experimentation, joint work with P. Dłotko and M. Mrozek.