Parallelohedra and
a Walk Around the Voronoi Conjecture

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In 1885 E. Fedorov introduced one of the most basic notions in Geometry of Numbers, the parallelohedron. Fedorov required that a parallelohedron satisfies the restriction that it fits together with a selection of translates of itself, to form a facet-to-facet tiling.

By definition, a d-dimensional *parallelohedron* P is a convex polyhedron to tile Euclidean d-space in face-to-face manner by its parallel copies. The centers of parallelohedra in a face-to-face tiling necessarily form a point lattice.

H. Minkowski (1897) proved two celebrated theorems on parallelohedra (necessary conditions and a sharp upper bound for the number of facets). Much later (1954), B. Venkov proved that necessary conditions by Minkowski are also sufficient. G. Voronoi (1908) developed a deep theory of parallelohedra of a special kind, the so-called Dirichlet-Voronoi parallelohedra. These parallelohedra, presently called Voronoi parallelohedra, are defined as the Voronoi cells for point lattices. Moreover, Voronoi conjectured that an arbitrary parallelohedron has the same structure as some appropriated Voronoi parallelohedron. Namely, he stated a conjecture on affine equivalence of an arbitrary parallelohedron to some Voronoi parallelohedron. Voronoi himself certified his conjecture for primitive parallelohedra (1908). Since then, regardless serious progress attained in the Voronoi conjecture in the first part of the last century (O. Zhitomirski, 1929), the conjecture remains unsolved.

We are going to present a survey of central results and some recent results in the theory of parallelohedra including certain thoughts relevant to the Voronoi conjecture.

This talk contains in part results of *Team Voronoi* (R. Erdahl, A. Garber, A. Gavrilyuk, A. Magazinov, M.Kozachok, Queen's University of Kingston, Steklov Institute, Moscow State University) obtained during its stay at the Fields Institute. The team thanks the Fields Institute and the organizers of the Discrete Geometry Program for their hospitality.