Abstract regular polytopes

String C-groups

A string C-group is a group of the form $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle$, whose generators $\boldsymbol{r}_j$ satisfy $(\boldsymbol{r}_j \boldsymbol{r}_k)^{p_{jk}} = \boldsymbol{e}$, with

$$p_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 2, & \text{if } |j-k| \geqslant 2. \end{cases}$$

Thus the $\boldsymbol{r}_j$ are involutions. Moreover, $\boldsymbol{G}$ satisfies the intersection property

$$\langle \boldsymbol{r}_i \mid i \in \mathsf{J} \rangle \cap \langle \boldsymbol{r}_i \mid i \in \mathsf{K} \rangle = \langle \boldsymbol{r}_i \mid i \in \mathsf{J} \cap \mathsf{K} \rangle,$$

for each $\mathsf{J}, \mathsf{K} \subseteq \mathsf{M} := \{0, 1, \ldots, m-1\}$.

If there are no other relations on the $\boldsymbol{r}_j$, then we have the Coxeter group $[p_1, \ldots, p_{m-1}]$, where we write $p_j := p_{j-1,j}$ for each $j = 1, \ldots, m-1$. Here, we usually assume that $p_j \geqslant 3$ for each j, to avoid degenerate cases.

Regular polytopes

We identify an (abstract) regular m-polytope $\mathcal{P}$ with its (automorphism) group, which is a string C-group $\boldsymbol{G}$ as before. The $\boldsymbol{r}_j$ are the canonical generators. We also call m the rank of $\mathcal{P}$.

A distinguished subgroup of $\boldsymbol{G}$ is one of the form

$$\boldsymbol{G}_{\mathsf{K}} := \langle \boldsymbol{r}_j \mid j \notin \mathsf{K} \rangle,$$

for some $\mathsf{K} \subseteq \mathsf{M}$.

For each $j \in \mathsf{M}$, the j-faces of $\mathcal{P}$ are the right cosets of the distinguished subgroup $\boldsymbol{G}_j := \boldsymbol{G}_{\{j\}}$. The incidence relation between faces is given by

$$\boldsymbol{G}_j \boldsymbol{a} \leqslant \boldsymbol{G}_k \boldsymbol{b} \iff j \leqslant k \text{ and } \boldsymbol{G}_j \boldsymbol{a} \cap \boldsymbol{G}_k \boldsymbol{b} \neq \emptyset.$$

Set $\boldsymbol{G}_{-1} = \boldsymbol{G}_m := \boldsymbol{G}$; then $\mathcal{P}$ is a partially ordered set with unique minimal and maximal elements.

Cofaces and sections

If $\boldsymbol{G}_{j-1}\boldsymbol{a} \leqslant \boldsymbol{G}_k\boldsymbol{b}$, then

$$\{\boldsymbol{F} \in \mathcal{P} \mid \boldsymbol{G}_{j-1}\boldsymbol{a} \leqslant \boldsymbol{F} \leqslant \boldsymbol{G}_k\boldsymbol{b}\}$$

is called a (j,k)-section; its rank is $k-j$. If $k=m$, then we obtain a $(m-j)$-coface.

Faces of rank j are called vertices, edges, ridges and facets for $j = 0, 1, m-2$ and $m-1$, respectively. Cofaces of rank j are called vertex-figures and edge-figures for $j = m-1$ and $m-2$.

Flatness

We call a regular polytope (or apeirotope) $\mathcal{P}$ (combinatorially) flat if every vertex of $\mathcal{P}$ is incident with every facet. Flatness occurs quite often, even in geometric regular polytopes.

The most important property of flatness is the following.

Theorem

Let $\mathcal{P}$ be a regular polytope. If the vertex-figure or facet of $\mathcal{P}$ is flat, then $\mathcal{P}$ itself is flat.

Corollary

If any proper section of a regular polytope is flat, then the polytope itself is flat.

Proof.

Suppose that the vertex-figure of $\mathcal{P}$ is flat; the proof for flat facets is just the dual argument. Let $\mathcal{V}$ be any vertex and $\mathcal{F}$ any facet. Choose some sequence

$$\mathcal{V} = \mathcal{V}_0 < \mathcal{E}_1 > \mathcal{V}_1 < \cdots < \mathcal{E}_k > \mathcal{V}_k$$

of incident vertices and edges such that $\mathcal{V}_k < \mathcal{F}$, and let $\mathcal{Q}_j$ be the vertex-figure of $\mathcal{P}$ at $\mathcal{V}_j$ for each j. Then

$$\mathcal{E}_k, \mathcal{F} \in \mathcal{Q}_k \implies \mathcal{E}_k < \mathcal{F} \implies \mathcal{V}_{k-1} < \mathcal{F}.$$

Induction on k now completes the proof. $\square$

Remark

In the geometric context, it is usually flat vertex-figures which induce flatness of the whole polytope.

Diagonals

Pairs of vertices of $\mathcal{P}$ are called diagonals. These fall into diagonal classes under the action of G, and thus each such class is represented by a pair of the form $\{G_0, G_0 a\}$, for some $a \in G$.

In some contexts, we need to distinguish between symmetric and asymmetric diagonals. For the former, the ordered pairs $(G_0, G_0 a)$ and $(G_0 a, G_0)$ are equivalent under G. Thus there is some $b \in G$ such that

$$(G_0, G_0 a)b = (G_0 a, G_0) \iff a^{-1} \in G_0 a G_0,$$

as can be seen by eliminating b.

It follows from this that diagonal classes (including the trivial one) correspond to unions $G_0 a G_0 \cup G_0 a^{-1} G_0$ of double cosets of G_0.

Regular pre-polytopes

A string group generated by involutions (sggi) is defined just as a string C-group, with the omission of the intersection property. An sggi $\boldsymbol{G}$ then determines a regular pre-polytope $\mathcal{P}$ in exactly the expected way.

Various criteria satisfied by an sggi make it a string C-group, of which the most important is

Theorem

An sggi $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle$ is a string C-group if and only if its distinguished subgroups $\boldsymbol{G}_0, \boldsymbol{G}_{m-1}$ are themselves string C-groups, and are such that

$$\boldsymbol{G}_0 \cap \boldsymbol{G}_{m-1} = \boldsymbol{G}_{0,m-1}.$$

Quotients

With $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle$ as before, we adopt the convention $\boldsymbol{r}_j := \boldsymbol{e}$ whenever $j \geqslant m$.

Let $\mathcal{Q}$ be a regular k-polytope, with group $\boldsymbol{H} = \langle \boldsymbol{s}_0, \ldots, \boldsymbol{s}_{k-1} \rangle$. If the mapping $\boldsymbol{r}_j \mapsto \boldsymbol{s}_j$ (for $j = 0, \ldots, m-1$) induces a homomorphism $\Phi \colon \boldsymbol{G} \to \boldsymbol{H}$, then we write $\mathcal{Q} := \mathcal{P}\Phi$, and call $\mathcal{Q}$ a quotient of $\mathcal{P}$. Thus we must have $k \leqslant m$, but strict inequality is allowed.

The numbers p_j determine the Schläfli type $\{p_1, \ldots, p_{m-1}\}$ of $\mathcal{P}$. We use the same symbol to denote the universal regular polytope, whose automorphism group is the Coxeter group $[p_1, \ldots, p_{m-1}]$. Thus the group of a regular polytope of Schläfli type $\{p_1, \ldots, p_{m-1}\}$ is a quotient of $[p_1, \ldots, p_{m-1}]$.

Quotient criteria

More generally, we may allow one or other of $\boldsymbol{G}, \boldsymbol{H}$ to be only an sggi, and ask for criteria in terms of Φ or $\boldsymbol{N} := \ker \Phi$ which ensure that it is a string C-group.

Theorem

If $\boldsymbol{G}$ is a string C-group and $\boldsymbol{N} \cap \boldsymbol{G}_0 \boldsymbol{G}_{m-1} = \{\boldsymbol{e}\}$, so that $\boldsymbol{N}$ is sparse, then $\boldsymbol{H}$ is a string C-group.

Theorem

If $\boldsymbol{H}$ is a string C-group, $\Phi \cap \boldsymbol{G}_0 = \{\boldsymbol{e}\}$, $\boldsymbol{N} \leqslant \boldsymbol{G}_{m-1}$ is such that $\boldsymbol{G}_{m-1}/\boldsymbol{N}$ is itself a C-group, then $\boldsymbol{G}$ is a string C-group.

Collapsing

We call the regular polytope $\mathcal{P}$ k-collapsible if $\langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{k-1} \rangle$ is a quotient of $\boldsymbol{G}$ under the mapping induced by $\boldsymbol{r}_j \mapsto \boldsymbol{e}$ for $j = k, \ldots, m-1$. This concept plays an important rôle in realization theory.

If we denote by $\boldsymbol{N}_k^+$ the normal closure of $\langle \boldsymbol{r}_k, \ldots, \boldsymbol{r}_{m-1} \rangle$ in $\boldsymbol{G}$, then the condition for k-collapsibility is

$$\langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{k-1} \rangle \cap \boldsymbol{N}_k^+ = \{\boldsymbol{e}\}.$$

Remark

The condition is also known as the flat amalgamation property with respect to k-faces. An equivalent condition to the above is

$$\boldsymbol{G} = \boldsymbol{N}_k^+ \rtimes \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{k-1} \rangle.$$

Central symmetry

We call the regular polytope $\mathcal{P}$ centrally symmetric if there is a central involution $z \in \boldsymbol{G}$ which fixes no vertex.

Theorem

If $\mathcal{P}$ is a centrally symmetric regular m-polytope such that, for every $j \leqslant m-2$, each j-face $\mathcal{G} < \mathcal{P}$ is determined by its vertex-set $\operatorname{vert}\mathcal{G}$, then the quotient $\boldsymbol{G}/\langle z\rangle$ is a C-group. It is therefore the automorphism group of a regular polytope, which is denoted $\mathcal{P}/2$.

Remark

Observe that $\mathcal{P}/2$ cannot be polytopal if $z \notin \boldsymbol{G}_j$ for $j = 0, m-1$, but $z \in \boldsymbol{G}_0\boldsymbol{G}_{m-1}$. On the other hand, if $z \notin \boldsymbol{G}_0\boldsymbol{G}_{m-1}$, then $\langle z\rangle$ is sparse, so that the quotient is a C-group.

Presentations

With each element $\boldsymbol{g} = \boldsymbol{r}_{j(1)} \cdots \boldsymbol{r}_{j(r)} \in \boldsymbol{G}$ is associated an index sequence $\mathsf{J} = \mathsf{J}(\boldsymbol{g}) := j(1) \ldots j(r)$ (not unique, of course). The index sequence thus ignores the particular labels given to the generators. Index sequences correspond to edge-paths, with each occurrence of 0 giving an edge. Similarly, to an edge-circuit corresponds an index cycle, and hence a relator in $\boldsymbol{G}$.

We then have the circuit criterion.

Theorem

The group $\boldsymbol{G}$ of a regular polytope is determined by the group of its vertex-figure and its edge-circuits.

If $\mathcal{P}, \mathcal{Q}$ are regular polytopes, with $\mathcal{Q} = \mathcal{P}\Phi$ a quotient of $\mathcal{P}$, and $\boldsymbol{N} = \ker \Phi = \langle \boldsymbol{n}_1, \ldots, \boldsymbol{n}_k \rangle$, say, let J_i be an index cycle associated with $\boldsymbol{n}_i$ for $i = 1, \ldots, k$. Then we write

$$\mathcal{Q} := \mathcal{P} / \langle\!\langle \mathsf{J}_1, \ldots, \mathsf{J}_k \rangle\!\rangle.$$

We have special notation for certain edge-circuits which are regular polygons, whose groups have canonical (involutory) generators $\boldsymbol{s}, \boldsymbol{t}$, say; we give the index sequences J, K of these generators.

For the Petrie polygon,

$$\mathsf{J} = 024\ldots, \quad \mathsf{K} = 135\ldots,$$

and for the deep hole,

$$\mathsf{J} = 0, \quad \mathsf{K} = 123\ldots(m-1)(m-2)\ldots21.$$

A Petrie polygon $\mathcal{C}$ of a regular m-polytope $\mathcal{P}$ has the following recursive definition: each successive $m-1$ edges of $\mathcal{C}$ are edges of some facet of $\mathcal{P}$, but no m successive edges are. In other words, $\mathcal{C}$ goes along some $m-1$ edges of a facet $\mathcal{F}$, say, and then departs to the (unique) facet $\mathcal{F}'$ which meets $\mathcal{F}$ on the ridge containing the previous $m-2$ edges of $\mathcal{C}$.

If $\mathcal{P}$ is determined solely by its Schläfli type $\{p_1, \ldots, p_{m-1}\}$ and length s of its Petrie polygon or t of its deep hole, then we write

$$\{p_1, \ldots, p_{m-1} : s\} := \{p_1, \ldots, p_{m-1}\}/\langle\langle (\mathsf{JK})^s \rangle\rangle,$$
$$\{p_1, \ldots, p_{m-1} \mid t\} := \{p_1, \ldots, p_{m-1}\}/\langle\langle (\mathsf{JK})^t \rangle\rangle,$$

respectively, with J, K as just defined.

We shall shortly see more elaborate notation for polyhedra.

Cubic toroids

$\{4, 3^{m-2}, 4\}$ is the abstract $(m+1)$-apeirotope corresponding to the tiling of $\mathbb{E}^m$ by cubes, with vertex-set $\mathbb{Z}^m$. Let $r \geqslant 2$. If we identify vertices of the cubic tiling by the sublattice generated by all re_j for $j = 1, \ldots, m$, then we obtain the cubic toroid

$$\{4, 3^{m-2}, 4 \mid r\} = \{4, 3^{m-2}, 4\}_{(r, 0^{m-1})}.$$

Similarly, identification by the sublattice generated by the points $r(\pm 1, \ldots, \pm 1)$ yields

$$\{4, 3^{m-2}, 4 : rm\} = \{4, 3^{m-2}, 4\}_{(r^m)}.$$

Remark

The third cubic toroid $\{4, 3^{m-2}, 4\}_{(r, r, 0^{m-2})}$ (for $m \geqslant 3$) does not admit such a simple expression.

Polyhedra

For polyhedra (the case $m = 3$), we have extra notation. For the k-zigzag, the index strings of the generators are

$$\mathsf{J} = 02, \quad \mathsf{K} = (12)^{k-1}1;$$

for the k-hole, they are

$$\mathsf{J} = 0, \quad \mathsf{K} = (12)^{k-1}1.$$

Thus the 2-hole is the deep hole, usually referred to just as the hole.

For a polyhedron $\mathcal{P}$ of Schläfli type $\{p, q\}$ determined by k-zigzags of length s_k and k-holes of length t_k, we write

$$\mathcal{P} = \{p, q : s_1, s_2, \ldots \mid t_2, t_3, \ldots\}.$$

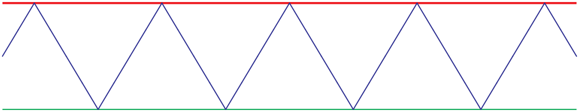
An unnecessary s_k or t_k is replaced by $\cdot$, and each sequence terminates with the last needed entry.

An interesting case

Theorem

For each $q \geqslant 3$ and $r \geqslant 2$,

$$\{3, q \mid \cdot, r\} = \{3, q : 2r\}.$$



Corollary

The Petrie polygons of the Platonic polyhedra are as follows:

$$\begin{aligned} \{4\}, &\quad \textit{for } \{3,3\}, \\ \{6\}, &\quad \textit{for } \{3,4\}, \\ \{10\}, &\quad \textit{for } \{3,5\}. \end{aligned}$$