Full rank:
classical polytopes

All dimensions

There are three classical regular d-polytopes for all dimensions d; for $d \geqslant 5$, these are the only ones. These polytopes are

- the d-simplex $\{3^{d-1}\}$, with $d+1$ vertices and group the symmetric group $\mathbf{A}_d \cong \boldsymbol{S}_{d+1}$ of order $(d+1)!$,
- the d-cross-polytope $\{3^{d-2}, 4\}$, with $2d$ vertices $\pm e_j$ for $j = 1, \ldots, d$ and group $\mathbf{C}_d$ of order $2^d d!$,
- its dual the d-cube $\{4, 3^{d-2}\}$, with 2^d vertices $(\pm 1, \ldots, \pm 1)$ and the same symmetry group $\mathbf{C}_d$.

Remark

In most dimensions, there is no neat expression for the vertex-set of the simplex. However, in the hyperplane

$$\mathbb{H}^d := \{(\xi_0, \ldots, \xi_d) \in \mathbb{E}^{d+1} \mid \xi_0 + \cdots + \xi_d = 0\},$$

we can take all permutations of $(d, (-1)^d)$.

Petrie polygons

We can easily work out the types of the Petrie polygons from the recursive definition.

For the d-simplex with vertices $a_0, \ldots, a_d$, say, a Petrie polygon of one facet has successive vertices $a_0, \ldots, a_{d-1}$. This polygon can now only go to a_d, and so the Petrie polygon $\mathcal{C}$ has length $d+1$. Since the product of the generating reflexions has single fixed point o, it follows that the type of $\mathcal{C}$ must be

$$\left\{\frac{d+1}{1, 2, \ldots, m}\right\},$$

with $m := \lfloor \frac{1}{2}(d+1) \rfloor$.

For the d-cross-polytope with vertices $\pm e_1, \ldots, \pm e_d$, a Petrie polygon of one facet has successive vertices $e_1, \ldots, e_d$. This polygon can now only go to $-e_1$. There are two consequences. First, the Petrie polygon $\mathcal{C}$ must have length $2d$. Second, going d steps along $\mathcal{C}$ is the inversion $-I$ in the origin, the single fixed point of the product of the generating reflexions. Since therefore the entries in the denominator of the generalized fraction must be odd, it follows that the type of $\mathcal{C}$ is

$$\left\{\frac{2d}{1, 3, \ldots, 2m-1}\right\},$$

with $m := \lfloor \frac{1}{2}(d+1) \rfloor$ as before.

The Petrie polygon of the d-cube has to be of the same type; this can be shown independently of the previous argument.

4-dimensions

Since we have already treated the plane and ordinary space, we are left with $\mathbb{E}^4$.

Before moving on the the 24-cell, we remark that, just as the vertex-set of the 3-cube $\{4, 3\}$ contains two copies of that of the tetrahedron $\{3, 3\}$ – actually, here

$$\{3, 3\} = \{4, 3\}^{\eta}$$

– so the vertex-set of the 4-cube $\{4, 3, 3\}$ contains two copies of that of the 4-cross-polytope $\{3, 3, 4\}$.

The inscribed cross-polytopes do not have full symmetry; instead, each has induced group $\mathbf{B}_4$ of index 2 in $\mathbf{C}_4$.

The 24-cell $\{3, 4, 3\}$

There are several different constructions of the 24-cell. Since quaternions provide such a nice approach to some 4-dimensional problems, for our purposes we initially take the 24-cell to have vertex-set $2\widehat{\mathbf{A}}$, with $\widehat{\mathbf{A}}$ the binary tetrahedral group of order 24.

Thus, as coordinate vectors, the vertices are all permutations of

$$(\pm 2, 0, 0, 0), \quad (\pm 1, \pm 1, \pm 1, \pm 1),$$

with all changes of sign. We immediately see that, among the vertices of $\{3, 4, 3\}$ are those of the cross-polytope and cube.

For the cube, this is explained by

$$\{4, 3, 3\} = \{3, 4, 3\}^{\eta}.$$

Since $\widehat{\mathbf{A}}$ is a subgroup of the binary octahedral group $\widehat{\mathbf{C}}$ of index 2, another choice of vertex-set for $\{3, 4, 3\}$ is $\sqrt{2}\widehat{\mathbf{B}}$, with $\widehat{\mathbf{B}}$ the other coset of $\widehat{\mathbf{A}}$ in $\widehat{\mathbf{C}}$. As coordinate vectors, these vertices are all permutations of

$$(\pm 1, \pm 1, 0, 0),$$

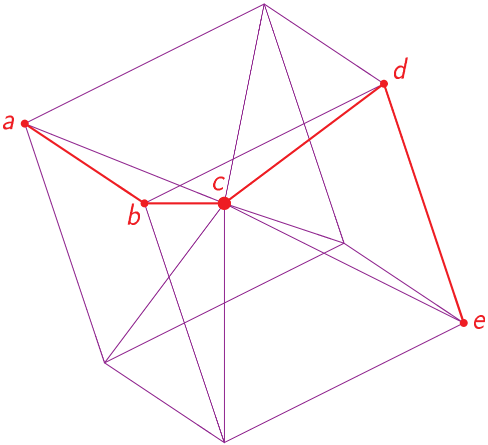
with all changes of sign.

This expression for the vertex-set shows that the vertices of $\{3, 4, 3\}$ are the mid-points of the edges of some cross-polytope $\{3, 3, 4\}$.

Petrie polygon

Since the vertex-figure of $\{3, 4, 3\}$ is the centrally symmetric cube $\{4, 3\}$, it follows that its edge-graph contains diametral planar polygons. In fact, these are hexagons: a typical one has vertices $(2, 0, 0, 0)$, $(\pm 1, 1, 1, 1)$ and their opposites.

If a, b, c, d, e are successive vertices of a Petrie polygon of $\{3, 4, 3\}$, then it is not hard to see that a, b, d, e are successive vertices of the (broad) vertex-figure of $\{3, 4, 3\}$ at c. Thus a, c, e are successive vertices of a diametral hexagon of $\{3, 4, 3\}$, which implies that the Petrie polygon itself must be of type $\{\frac{12}{s,t}\}$ for some s, t. Since two steps along it give a planar hexagon, we must have $s, t \equiv \pm 1 \pmod 6$, and it follows that $s = 1, t = 5$, giving the Petrie polygon of type $\{\frac{12}{1,5}\}$. See the picture following.


a
b
c
d
e

Realizations of the 24-cell

We begin, of course, with $P_3 = \{3, 4, 3\}$ itself of dimension $d_3 = 4$, which shows that the layer vector is $\Lambda = (1, 8, 6, 8, 1)$. Observe next that $\{3, 4, 3\}$ is 2-collapsible onto its face $\{3\}$; thus $\{3, 4, 3\}/2$ will have a pure component $P_1 = \{3\}$ of dimension $d_1 = 2$. This is enough information to describe the entire realization space, because (apart from the trivial realization $P_0 = \{1\}$ of dimension $d_0 = 1$) $\{3, 4, 3\}/2$ must then have just one other pure component P_2 of dimension $d_2 = 12 - 1 - 2 = 9$, leaving a last faithful pure realization P_4 of dimension $d_4 = 12 - 4 = 8$.

For the corresponding cosine vectors, we initially have

$$\Gamma_0 = (1, 1, 1, 1, 1),$$
$$\Gamma_1 = (1, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, 1),$$
$$\Gamma_3 = (1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1).$$

We then calculate Γ_2, Γ_4 by

$$\Gamma_2 = \frac{1}{9}\big(12(1,0,0,0,1) - \Gamma_0 - 2\Gamma_1\big) = (1, 0, -\tfrac{1}{3}, 0, 1),$$
$$\Gamma_4 = \frac{1}{8}\big(12(1,0,0,0,-1) - 4\Gamma_3\big) = (1, -\tfrac{1}{4}, 0, \tfrac{1}{4}, -1),$$

leading to the cosine matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 1 & -\frac{1}{2} & 1 \\ 1 & 0 & -\frac{1}{3} & 0 & 1 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 1 & -\frac{1}{4} & 0 & \frac{1}{4} & -1 \end{bmatrix}$$

.

We can identify the vertices of P_2 as those of three regular tetrahedra in mutually orthogonal 3-dimensional subspaces of $\mathbb{E}^9$.

So far, we have not mentioned products. We easily find that

$$\begin{aligned} \Gamma_3^2 &= \tfrac{1}{4}\Gamma_0 + \tfrac{3}{4}\Gamma_2, \\ \Gamma_1\Gamma_3 &= \Gamma_4. \end{aligned}$$

In other words, P_2 is the non-trivial component of $P_3 \otimes P_3$, while $P_4 = P_1 \otimes P_3$.

Notice as well that $P_1 \otimes P_2 = P_2$.

The 600-cell $\{3, 3, 5\}$

We can take the vertices of the (convex) 600-cell $P_5 := \{3, 3, 5\}$ in $\mathbb{E}^4$ to be $2\widehat{\mathbf{G}}$, with $\widehat{\mathbf{G}}$ the binary icosahedral group or icosians. As coordinate vectors, these are all even permutations with all changes of sign of

$$(2, 0, 0, 0), \quad (1, 1, 1, 1), \quad (\tau, 1, \tau^{-1}, 0),$$

where (as usual) $\tau := \frac{1}{2}(1 + \sqrt{5})$.

This formulation shows that the symmetry group $[3, 3, 5]$ consists of all mappings of the form

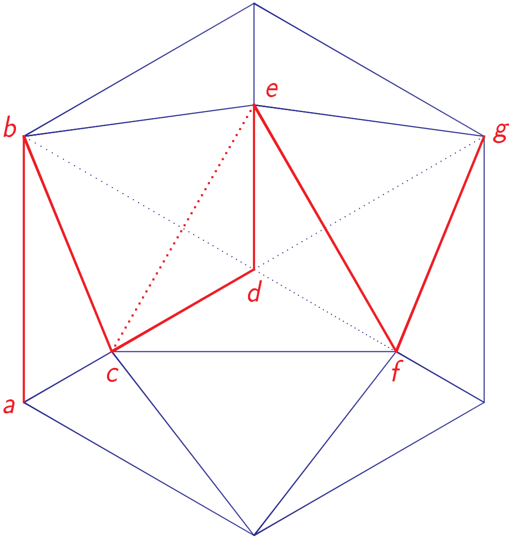
$$\mathbf{x} \mapsto \bar{\mathbf{a}}\mathbf{x}\mathbf{b} \quad \text{or} \quad \bar{\mathbf{a}}\,\bar{\mathbf{x}}\mathbf{b},$$

with $\mathbf{a}, \mathbf{b} \in \widehat{\mathbf{G}}$.

Petrie polygon

The 600-cell has planar diametral polygons, since the vertex-figure $\{3,5\}$ is centrally symmetric. These subtend an angle $\arccos(\tau/2) = \pi/5$ at the centre, and so are decagons $\{10\}$.

If a, b, c, d, e, f, g are successive vertices of a Petrie polygon of $\{3,3,5\}$, then a, b, c, e, f, g are successive vertices of a Petrie polygon of the (broad) vertex-figure of $\{3,3,5\}$ at d. Thus a, d, g are successive vertices of a diametral decagon of $\{3,3,5\}$, so that the Petrie polygon is of type $\{\frac{30}{s,t}\}$ for some $1 \leqslant s < t \leqslant 15$. Since three steps along give a planar decagon, we must have $s, t \equiv \pm 1 \pmod{10}$. Because of the possible double rotations in the group, $s, t \neq 9$; thus the Petrie polygon is of type $\{\frac{30}{1,11}\}$. See the following picture.


e
b
g
d
c
f
a

Star polytopes

The only possible non-integer entry in the Schläfli symbol of a 4-dimensional regular polytope is $\frac{5}{2}$. All the 4-dimensional star polytopes are related by duality and vertex-figure replacement. By the latter is meant replacing an existing vertex-figure by another with the same vertices and symmetry group.

Theorem

A suitable sequence of vertex-figure replacements and dualities applied to a regular star polytope P will result in a convex polytope Q. Then P will have the same vertices as Q or its dual Q^δ. Every regular star polytope is obtainable by reversing such processes.

In fact, the same idea holds in lower dimensions; however, in $\mathbb{E}^3$ we found the star polyhedra in a different way.

The 4-dimensional star polytopes

$$
\begin{array}{ccccc}
\{5,3,3\} & \{3,3,5\} & & & \\
 & \{3,5,\frac{5}{2}\} & \{\frac{5}{2},5,3\} & & \\
 & \{5,\frac{5}{2},5\} & & & \\
 & \{5,3,\frac{5}{2}\} & & \{\frac{5}{2},3,5\} & \\
 & & & \{\frac{5}{2},5,\frac{5}{2}\} & \\
 & & \{5,\frac{5}{2},3\} & \{3,\frac{5}{2},5\} & \\
 & & & \{3,3,\frac{5}{2}\} & \{\frac{5}{2},3,3\}
\end{array}
$$

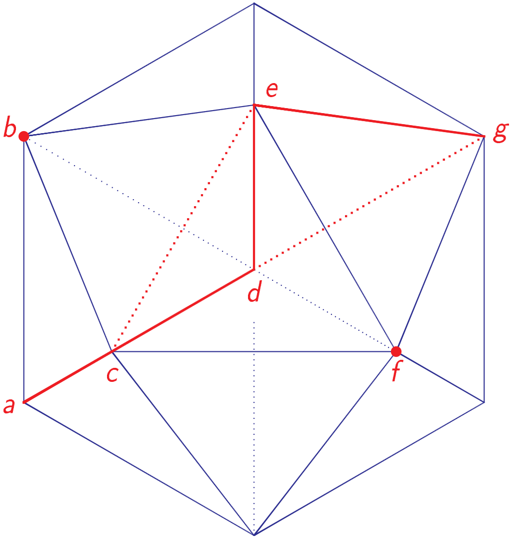
Duals are in the same row; vertex-figure replacements are in the same column.

Petrie polygons

If as before a, b, c, d, e, f, g are successive vertices of a Petrie polygon of $\{3,3,5\}$, then the two adjacent facets of $\{3,5,\frac{5}{2}\}$ which are the broad vertex-figures of $\{3,3,5\}$ at b, f meet on the trigon c, d, e. Then a, c, d, e, g are successive vertices of a Petrie polygon of $\{3,5,\frac{5}{2}\}$; thus two steps along it go one step along a diametral dodecagon. It follows that the Petrie polygon is of type $\{\frac{20}{1,9}\}$. See the following picture.

A similar argument shows that successive vertices of a Petrie polygon of $\{5,\frac{5}{2},5\}$ are alternate vertices of a Petrie polygon of $\{3,3,5\}$, so that the Petrie polygon here is of type $\{\frac{15}{1,4}\}$.

For $\{5,3,\frac{5}{2}\}$, whose Petrie polygon will be of the same type as that of its dual $\{\frac{5}{2},3,5\}$, we can argue that it is not altered by the interchange $5 \leftrightarrow \frac{5}{2}$, and so (of the permissible double rotations) it can only be of type $\{\frac{12}{1,5}\}$.


e
b
g
d
c
f
a

Realizations

Since $\{3,3,5\} \cong \mathbf{\{3,3,5\}}$ (the abstract polytope), we see that the layer vector is $\Lambda = (1,12,20,12,30,12,20,12,1)$. This gives initial cosine vector

$$\Gamma_5 = (1, \tfrac{\tau}{2}, \tfrac{1}{2}, \tfrac{\tau^{-1}}{2}, 0, -\tfrac{\tau^{-1}}{2}, -\tfrac{1}{2}, -\tfrac{\tau}{2}, -1).$$

Changing the sign of $\sqrt{5}$ gives the cosine vector

$$\Gamma_6 = (1, -\tfrac{\tau^{-1}}{2}, \tfrac{1}{2}, -\tfrac{\tau}{2}, 0, \tfrac{\tau}{2}, -\tfrac{1}{2}, \tfrac{\tau^{-1}}{2}, -1)$$

of the stellated 600-cell $P_6 := \{3,3,\tfrac{5}{2}\}$.

For the rest, we dispose of $\mathcal{Q} := \{3,3,5\}/2$ first. As a realization of $\mathcal{P}$, its simplex realization has cosine vector $(1, 0^7, 1)$.

The non-trivial component P_1 of $P_5 \otimes P_5$ has dimension $d_1 = 9$, with cosine vector Γ_1 given by $\Gamma_5^2 = \frac{1}{4}\Gamma_0 + \frac{3}{4}\Gamma_1$; thus,

$$\Gamma_1 = (1, \tfrac{1}{3}\tau, 0, -\tfrac{1}{3}\tau^{-1}, -\tfrac{1}{3}, -\tfrac{1}{3}\tau^{-1}, 0, \tfrac{1}{3}\tau, 1).$$

Similarly, $\Gamma_6^2 = \frac{1}{4}\Gamma_0 + \frac{3}{4}\Gamma_2$ gives pure component P_2, again with dimension $d_2 = 9$, and cosine vector

$$\Gamma_2 = (1, -\tfrac{1}{3}\tau^{-1}, 0, \tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}\tau, 0, \tfrac{1}{3}\tau^{-1}, 1).$$

Next, define P_3 of dimension $d_3 = 16$ by its cosine vector

$$\Gamma_3 := \Gamma_5\Gamma_6 = (1, -\tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, 0, -\tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, 1).$$

We wrap up $\mathcal{Q}$ by observing that it has 4 non-trivial diagonals, and so we only need one final pure realization P_4 of dimension $d_4 = 60 - 1 - 9 - 9 - 16 = 25$, whose cosine vector is given by

$$\begin{aligned}\Gamma_4 &= \tfrac{1}{25}\big(60(1, 0^7, 1) - \Gamma_0 - 9\Gamma_1 - 9\Gamma_2 - 16\Gamma_3\big)\\ &= (1, 0, -\tfrac{1}{5}, 0, \tfrac{1}{5}, 0, -\tfrac{1}{5}, 0, 1).\end{aligned}$$

Remark

We know that P_1, P_2 must be pure, because $\mathcal{P}$ has inscribed copies of $\{3, 4, 3\}$. Then P_3 must be pure, because – being invariant under the change of sign of $\sqrt{5}$ – having one of P_1, P_2 as a component would force the other. Last, P_4 is the only possibility to fill in the gap.

We similarly need 4 pure centrally symmetric realizations, of which we already know P_5, P_6.

Now $P_8 := P_1 \otimes P_6 = P_2 \otimes P_5$, with cosine vector

$$\Gamma_8 = (1, -\tfrac{1}{6}, 0, \tfrac{1}{6}, 0, -\tfrac{1}{6}, 0, \tfrac{1}{6}, -1),$$

has dimension at most $d_8 = 4 \cdot 9 = 36$.

We then consider $P_5 \otimes P_5 \otimes P_5$, with cosine vector

$$\Gamma_5^3 = (\tfrac{1}{4}\Gamma_0 + \tfrac{3}{4}\Gamma_1)\Gamma_5 = \tfrac{1}{4}\Gamma_5 + \tfrac{3}{4}\Gamma_1\Gamma_5,$$

whose dimension is at most 20. Hence $P_1 \otimes P_5$ cannot have a pure component other than P_5 of dimension more than 16.

Since we have $4+4+16+36=60$, we see that P_8 must be pure. Moreover, the final pure component P_7 does have dimension $d_7 = 16$, with cosine vector given by

$$\begin{aligned}\Gamma_7 &= \tfrac{1}{16}(60(1,0^7,-1) - 4\Gamma_5 - 4\Gamma_6 - 36\Gamma_8) \\ &= (1, \tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, -1).\end{aligned}$$

Bear in mind that the cross-polytope realization of $\mathcal{P}$ will have cosine vector $(1, 0^7, -1)$.

Observe that

$$\Gamma_1\Gamma_5 = \tfrac{1}{3}\Gamma_1 + \tfrac{2}{3}\Gamma_7, \quad \Gamma_2\Gamma_6 = \tfrac{1}{3}\Gamma_6 + \tfrac{2}{3}\Gamma_7.$$

Cosine matrix of $\{3, 3, 5\}$

In summary, the cosine matrix is

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1}{3}\tau & 0 & -\frac{1}{3}\tau^{-1} & -\frac{1}{3} & -\frac{1}{3}\tau^{-1} & 0 & \frac{1}{3}\tau & 1 \\
1 & -\frac{1}{3}\tau^{-1} & 0 & \frac{1}{3}\tau & -\frac{1}{3} & \frac{1}{3}\tau & 0 & -\frac{1}{3}\tau^{-1} & 1 \\
1 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 1 \\
1 & 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & -\frac{1}{5} & 0 & 1 \\
1 & \frac{1}{2}\tau & \frac{1}{2} & \frac{1}{2}\tau^{-1} & 0 & -\frac{1}{2}\tau^{-1} & -\frac{1}{2} & -\frac{1}{2}\tau & -1 \\
1 & -\frac{1}{2}\tau^{-1} & \frac{1}{2} & -\frac{1}{2}\tau & 0 & \frac{1}{2}\tau & -\frac{1}{2} & \frac{1}{2}\tau^{-1} & -1 \\
1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -1 \\
1 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & -1
\end{bmatrix}$$

The 120-cell

We shall not say as much about the dual $\{5,3,3\}$ to the 600-cell. However, once again, quaternions give a neat way to investigate it.

It is not too hard to see that one vertex of the geometric 120-cell $\{5,3,3\}$ dual to the $\{3,3,5\}$ with vertex set $2\widehat{\mathbf{G}}$ is the quaternion $\frac{1}{\sqrt{2}}(1+\mathbf{i})$. Since $\widehat{\mathbf{A}} \leqslant \widehat{\mathbf{G}}$ with index 5, and $\widehat{\mathbf{B}} \subset \widehat{\mathbf{C}}$ is the other coset of $\widehat{\mathbf{A}}$ in $\widehat{\mathbf{C}}$, we see that (with this initial vertex) the vertex-set of $\{5,3,3\}$ can be identified with $\widehat{\mathbf{G}}\widehat{\mathbf{B}}\widehat{\mathbf{G}}$.

We shall not list the vertices; the list is rather long. In any event, a suitable scaling is needed to make them look tidier.

A further remark is in order. Suppose that $\mathbf{p} \in \widehat{\mathbf{G}}$ is an element of order 5, for example $\mathbf{p} := \frac{1}{2}(-\tau + \mathbf{i} + \tau^{-1}\mathbf{j})$. Then the powers of $\mathbf{p}$ are the (left or right) coset representatives of $\widehat{\mathbf{A}}$ in $\widehat{\mathbf{G}}$.

We can now write

$$\mathrm{vert}\{5,3,3\} = \bigcup \{\mathbf{p}^{-r}\widehat{\mathbf{B}}\mathbf{p}^{s} \mid r, s = 0, \ldots, 4\}.$$

However, we may observe that $\mathbf{b}^{-1}\widehat{\mathbf{G}}\mathbf{b} = \widehat{\mathbf{G}}^{\dagger}$ for any $\mathbf{b} \in \widehat{\mathbf{B}}$ – recall that † changes the sign of $\sqrt{5}$. After a little manipulation, we find an alternative expression

$$\mathrm{vert}\{5,3,3\} = \bigcup \{(\mathbf{p}_r^{*}\mathbf{p}_r\widehat{\mathbf{G}} \mid r = 0, \ldots, 4\},$$

where $\mathbf{p}_r := \mathbf{p}^{r}$ and $\mathbf{x}^{*} := \overline{\mathbf{x}}^{\dagger}$ $(= (\mathbf{x}^{\dagger})^{-1})$.

Now the points $\mathbf{p}_r^* \mathbf{p}_r$ form the vertices of a regular 4-simplex, so the last relation expresses $\mathrm{vert}\{5,3,3\}$ either as 120 copies of the vertices of a 4-simplex, or as 5 copies of $\mathrm{vert}\{3,3,5\}$.

There follows a somewhat remarkable result.

Theorem

The vertex-set $\mathrm{vert}\{5,3,3\}$ of the 120-cell contains the vertex-sets of all the other classical regular 4-polytopes.

Since $\widehat{\mathbf{A}} \leqslant \widehat{\mathbf{G}}$, we have already captured the remaining four.