Operations and constructions

There is a wide range of operations on (abstract) groups which may yield C-groups. By and large, they fall into two main areas. First, in mixing we pick suitable involutory elements in a given group. Second, in twisting we augment a given group by means of suitable automorphisms.

However, these notions are not mutually exclusive, since we find it useful to allow inner automorphisms in twisting, and so, strictly speaking, such an operation is mixing.

Further, we prefer to regard certain operations as examples of constructions, particularly when the abstract and geometric part company. Thus there is no hard-and-fast line to be drawn between operations and constructions, and it is largely a matter of taste as to how we categorize them.

Mixing

Most of the mixing operations are applied to C-groups, to yield new closely related C-groups which may even be the same. The general notation for operations will be bold Greek uncials, to distinguish them from ordinary Greek letters used as (for example) real numbers. These will then apply to the corresponding regular polytopes as well.

A general principle is that, if a mixing operation is initially defined for regular m-polytopes, then subsequently the same operation is applied to the m-cofaces of polytopes of higher rank.

Duality

Strictly speaking, the duality operation

$$\delta\colon (s_0, \ldots, s_{m-1}) \mapsto (s_{m-1}, \ldots, s_0) =: (r_0, \ldots, r_{m-1})$$

is a mixing operation, and so fits in here. The notation $\mathcal{P} := \mathcal{Q}^{\delta}$ for the dual $\mathcal{P}$ with group $G(\mathcal{P}) = \langle r_0, \ldots, r_{m-1} \rangle$ of $\mathcal{Q}$ with group $G(\mathcal{Q}) = \langle s_0, \ldots, s_{m-1} \rangle$ conforms to the general pattern.

Remark

With duality, we find one of the sharpest divides between the abstract and geometric theories, since a geometric regular polytope need not have a geometric dual, at least in the same space, and may even have no geometric dual which exhibits its combinatorial structure at all.

Facetting

As an important case, the facetting operation φ_k applies to a polygon $\mathcal{Q}$ (of rank 2), and is given on the generators of its group $\boldsymbol{H} = \langle \boldsymbol{s}_0, \boldsymbol{s}_1 \rangle$ by

$$\varphi_k \colon (\boldsymbol{s}_0, \boldsymbol{s}_1) \mapsto ((\boldsymbol{s}_0 \boldsymbol{s}_1)^{k-1} \boldsymbol{s}_0, \boldsymbol{s}_1) =: (\boldsymbol{r}_0, \boldsymbol{r}_1),$$

to give a new C-group $\boldsymbol{G} = \boldsymbol{H}^{\varphi_k} := \langle \boldsymbol{r}_0, \boldsymbol{r}_1 \rangle$.

If $\mathcal{Q}$ is a q-gon with successive vertices $v_0, v_1, \ldots, v_{q-1}$ in cyclic order, then successive vertices of the new polygon $\mathcal{P} = \mathcal{Q}^{\varphi_k}$ are $v_0, v_k, v_{2k}, \ldots$. Thus we may assume that $1 \leqslant k \leqslant \frac{1}{2}q$. If $(q, k) = s > 1$, then $\mathcal{P}$ is a (q/s)-gon, reducing to a digon $\{2\}$ if $s = \frac{1}{2}q$.

Theorem

The facetting operations applied to a regular q-gon satisfy

$$\varphi_i \varphi_j = \varphi_k,$$

with $1 \leqslant k \leqslant \frac{1}{2}q$ such that $k \equiv \pm ij \pmod{q}$.

This is obvious from the geometric description. On the algebraic level, it is made more clear if we write $(s_0 s_1)^{k-1} s_0 = (s_0 s_1)^k s_1$, so that

$$((s_0 s_1)^i s_1 \cdot s_1)^j s_1 = (s_0 s_1)^{ij} s_1.$$

For the most part, we have $(q, k) = 1$, so that φ_k is invertible, but occasionally the case $(q, k) > 1$ plays a useful rôle.

Petrie operation

The Petrie operation π applies to a regular polyhedron $\mathcal{Q}$ (of rank 3), and is given on the generators of its group $H = \langle s_0, s_1, s_2 \rangle$ by

$$\pi : (s_0, s_1, s_2) \mapsto (s_0 s_2, s_1, s_2) =: (r_0, r_1, r_2),$$

to give a new group $G = H^\pi := \langle r_0, r_1, r_2 \rangle$, and corresponding polytope $\mathcal{P} := \mathcal{Q}^\pi$ (when it exists).

We see that π is involutory; we call $\mathcal{Q}^\pi$ the Petrial of $\mathcal{Q}^\pi$. Only on very rare occasions in rank 3 is G not a C-group, but it can fail to be in higher ranks, as we shall shortly see.

It is routine to verify

Theorem

In rank 3, duality δ and the Petrie operation π satisfy $(\pi\delta)^3 = \varepsilon$, the identity.

It is sometimes useful to employ the dual operation $\pi^* = \delta\pi\delta = \pi\delta\pi$ to π, which is given by

$$\pi^* \colon (s_0, s_1, s_2) \mapsto (s_0, s_1, s_0 s_2) =: (r_0, r_1, r_2).$$

We can think of π^* as the conjugate of π under δ.

The following is obvious.

Theorem

The Petrie operation π and facetting operation φ_k commute.

We shall write $\pi_k := \varphi_k\pi = \pi\varphi_k$ for their composition.

The most general circumstance under which the Petrie operation breaks down in higher rank is given by

Theorem

Let $H = \langle s_0, \ldots, s_{m-1} \rangle$ be a string C-group with $m \geqslant 4$. If $H_{m-2,m-1} = \langle s_0, \ldots, s_{m-3} \rangle$ has a relator which contains s_{m-3} an odd number of times, then H^π is not a C-group.

Proof.

Let h be the relator (so that h is a word in $s_0, \ldots, s_{m-3}$ such that $h = e$), and suppose that h contains s_{m-3} an odd number k of times. Let g be the element obtained by replacing s_{m-3} by $r_{m-3} = s_{m-3}s_{m-1}$ (and s_j by r_j for other j). Then

$$g = hs_{m-1}^k = es_{m-1} = s_{m-1}.$$

In $G = H^\pi$ we thus have $r_{m-1} \in \langle r_0, \ldots, r_{m-3} \rangle$, violating the intersection property. $\square$

Halving

The halving operation η initially applies to a regular polyhedron $\mathcal{Q}$ with tetragonal 2-faces, and is

$$\eta\colon (s_0, s_1, s_2) \mapsto (s_0 s_1 s_0, s_2, s_1) =: (r_0, r_1, r_2),$$

to give a new group $G = H^\eta := \langle r_0, r_1, r_2 \rangle$, and corresponding polytope $\mathcal{P} := \mathcal{Q}^\eta$.

If $\mathcal{Q}$ is of Schläfli type $\{4, q\}$, then $\mathcal{P} := \mathcal{Q}^\eta$ is a self-dual regular polyhedron of type $\{q, q\}$, with duality induced by conjugation by s_0.

Theorem

If $\mathcal{Q}$ has an odd edge-circuit, then $\mathcal{Q}^\pi$ has the same vertices and group as $\mathcal{Q}$. If not, then the number of vertices and group order are halved.

Petrie contraction

In a vague sense, Petrie contraction is related to the dual of the Petrie operation. Formally, it is the mixing operation ϖ on the group $\boldsymbol{H} = \langle \boldsymbol{s}_0, \ldots, \boldsymbol{s}_m \rangle$ of a regular polytope $\mathcal{Q}$ given by

$$\varpi \colon (\boldsymbol{s}_0, \ldots, \boldsymbol{s}_m) \mapsto (\boldsymbol{s}_1, \boldsymbol{s}_0\boldsymbol{s}_2, \boldsymbol{s}_3, \ldots, \boldsymbol{s}_m) =: (\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1}).$$

Thus we write $\boldsymbol{G} := \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle = \boldsymbol{H}^{\varpi}$, and $\mathcal{P} := \mathcal{Q}^{\varpi}$ for the corresponding regular polytope, if it exists. Observe that ϖ reduces rank by 1.

Under many circumstances, $\boldsymbol{G} = \boldsymbol{H}$; the alternative cases are less interesting, but still often worth noting. We shall postpone giving examples until they are relevant.

Twisting

A twisting operation applies one or more (usually) involutory automorphisms $\tau_1, \ldots$ to an existing (usually) C-group $\boldsymbol{H} = \langle \boldsymbol{s}_1, \ldots \rangle$, to yield a new string C-group $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle$. A similar notation to that for mixing can be employed:

$$(\boldsymbol{s}_1, \ldots, \tau_1, \ldots) \mapsto (\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1}),$$

where each $\boldsymbol{r}_j$ is an $\boldsymbol{s}_k$ or a τ_k.

We specifically allow inner rather than outer automorphisms, in which case we genuinely have a mixing operation. Moreover, when applied to a Coxeter diagram, a twist τ is proper if the unit normals to the reflexion hyperplanes of the diagram can be chosen so that τ permutes them (that is, does not change any of their signs); otherwise τ is improper (and then usually inner).

We shall leave to the appropriate place most examples of twisting. However, there is one striking example which is worth presenting here.

Suppose that $\boldsymbol{H} = \boldsymbol{C}_2^m = \boldsymbol{C}_2 \times \cdots \times \boldsymbol{C}_2$, the elementary abelian group of order 2^m, with involutory generators $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m$. Let $\boldsymbol{S}_m = \langle \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{m-1} \rangle$ be the symmetric group on $\{1, \ldots, m\}$, with $\boldsymbol{\tau}_j := (j\ j{+}1)$ for $j = 1, \ldots, m-1$. Then

$$(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{m-1}) \mapsto (\boldsymbol{s}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{m-1}) =: (\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1}),$$

with the $\boldsymbol{\tau}_j$ acting as indicated on the indices of the $\boldsymbol{s}_k$, gives the automorphism group $\boldsymbol{G} := \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle$ of the (abstract) m-cube $\{4, 3^{m-2}\}$.

Constructions

By definition, constructions are usually geometric in nature, since it is often (but not invariably) unclear what an appropriate abstract analogue would look like.

Typically, a construction will modify one or more generating reflexions of a given string C-group, or adjoin a new reflexion to it. Note that such constructions do not always lead to C-groups.

We begin with an important observation.

Theorem
If $S, T_1, \ldots, T_k$ are linear reflexions such that $S \rightleftharpoons T_j$ for each $j = 1, \ldots, k$, then

$$S \rightleftharpoons T_1 \cap \cdots \cap T_k,$$

with the usual identification of geometric reflexions with their mirrors.

Now let $\mathbf{H} := \langle S_0, \ldots, S_{m-1} \rangle$ be a given geometric C-group, acting on the euclidean space $\mathbb{E}$, say. For each j, define

$$K_k := S_k \cap S_{k+1} \cap \cdots \cap S_{m-1}.$$

Using the previous observation, as a reflexion $K_k \rightleftharpoons S_j$ for each $j \neq k-1$, but K_k does not commute with S_{k-1}.

Hence, if we define the operation

$$\begin{aligned} \boldsymbol{\kappa}_{jk} \colon (S_0, \ldots, S_{m-1}) \mapsto (S_0, \ldots, S_{j-1}, S_j K_k, S_{j+1}, \ldots, S_{m-1}) \\ =: (R_0, \ldots, R_{m-1}), \end{aligned}$$

then we cannot obtain a string C-group $\mathbf{G} := \langle R_0, \ldots, R_{m-1} \rangle$ when $k \geqslant 1$ unless $j = k-2$ or k. Since the case $j = k$ is particularly important, we abbreviate $\boldsymbol{\kappa}_k := \boldsymbol{\kappa}_{kk}$.

There are several special cases, beginning with $k = 0$. If $\mathbf{H}$ is the symmetry group of an apeirotope, then $K_0 = \emptyset$, so that $\boldsymbol{\kappa}_0$ is not defined. Thus only the case of polytopes is of interest, and here we adopt the notation $Z := K_0$; thus Z is inversion in the centre of the corresponding (finite) regular polytope Q.

For (finite) regular polytopes, we write $\zeta_j := \boldsymbol{\kappa}_{j0}$, which replaces S_j by $S_j Z$, and then further abbreviate $\zeta := \zeta_0$. This is an operation of central importance.

Remark

Unfortunately, there are rare occasions when Q is a polytope, but $P = Q^\zeta$ is not. This means that the polytopality of P needs to be checked.

Faces

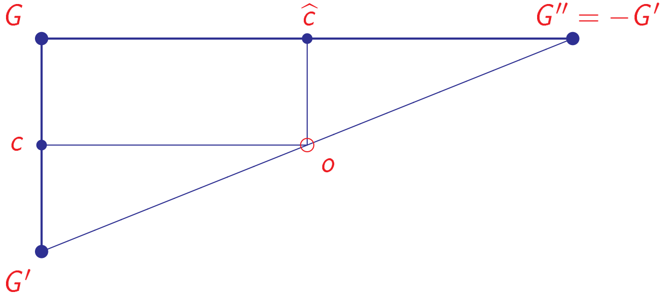
We shall need to know how faces behave under ζ.

Theorem

Let F be an initial (proper) face with centre c of a finite regular polytope P with centre o. Then, under ζ, the corresponding initial face $\widehat{F}$ of P^ζ with centre $\widehat{c}$ is as follows:

- *if F is not a blend with component $\{2\}$ and $c = o$, then $\widehat{F} = F^\zeta$ with $\widehat{c} = o$;*
- *if F is not a blend with component $\{2\}$ and $c \neq o$, then $\widehat{F} = F^\zeta \# \{2\}$ with $\widehat{c} = o$;*
- *if $F = G \# \{2\}$ is a blend with component $\{2\}$ and $c = o$, then $\widehat{F} = G^\zeta$ with $\widehat{c} \neq o$;*
- *if $F = G \# \{2\}$ is a blend with component $\{2\}$ and $c \neq o$, then $\widehat{F} = G^\zeta \# \{2\}$ with $\widehat{c} \neq o$ and $\langle c, \widehat{c} \rangle = 0$.*

The proof of the last part is basically on hand of this picture:



We often use the convenient shorthand

$$Q \diamond \{2\} := Q^{\zeta} \# \{2\},$$

so that the faces occurring above could be written $G \diamond \{2\}$.

An apeirotope construction

For apeirotopes, it is $\boldsymbol{\kappa} := \boldsymbol{\kappa}_1$ which is of most interest, and here there is an crucial extension of the notion. A natural convention is to take the initial vertex of the apeirotope Q to be the origin o, so that the group $\langle S_1, \ldots, S_{m-1} \rangle$ of the vertex figure is an orthogonal group. Now $K := K_1 = W$, the Wythoff space, so that the mirror of KS_1 is $W \oplus S_1^{\perp}$.

However, with a slight modification the construction still applies when Q is a finite polytope. In the usual cases of apeirotopes, W is a point-set. So, we define $P := Q^{\boldsymbol{\kappa}}$ as previously, but with $W = \{v\}$, where v is the initial vertex of Q. Now P may be an apeirotope of the same rank, which will be discrete just when Q is crystallographic. Indeed, if G is a proper face of Q, then the face of P of the same rank is $F := G^{\boldsymbol{\kappa}}$. We shall give examples of this construction later.

In terms of the group generators (whether the polytope is finite or not), we have $R_j := S_j$ for $j \neq 1$, and R_1 is defined by

$$xR_1 := 2v - xS_1.$$

An important consequence of the definition is the following.

Theorem

If G is a face of Q of rank $k \geqslant 2$, then the corresponding k-face of $P = Q^{\boldsymbol{\kappa}}$ is $F = G^{\boldsymbol{\kappa}}$.

Another consequence is

Theorem

If P, Q are two regular polytopes, then

$$(P \# Q)^{\boldsymbol{\kappa}} = P^{\boldsymbol{\kappa}} \# Q^{\boldsymbol{\kappa}}.$$

In the case of polygons, as before for $p \geqslant 2$ we define p'' by

$$\frac{1}{p} + \frac{1}{p''} = \frac{1}{2},$$

with the natural conventions $2'' = \infty$, $\infty'' = 2$. Then we have

Theorem

For each $p \geqslant 2$,

$$\{p\}^{\kappa} = \{p''\}.$$

Corollary

For a general regular polygon $\{p\} = \{p_1\} \# \cdots \# \{p_k\}$, with $\infty \geqslant p_1 > \cdots > p_k \geqslant 2$,

$$\{p\}^{\kappa} = \{p''\} := \{p_k''\} \# \cdots \# \{p_1''\}.$$

Vertex-figures

When κ is applied to an apeirotope P with vertex-figure Q, the vertex-figure of P^κ is (by definition) Q^ζ. In the discussion of ζ, we saw how blends with $\{2\}$ are interchanged with lower dimensional polytopes not having o as centre.

Of course, this interchange was the original motivation for applying κ to finite polytopes; here, the vertex-figure Q is a polytope whose centre is not the vertex v with respect to which ζ is being applied.

The free abelian apeirotope

Another important construction is that of the free abelian apeirotope. Let Q be a finite regular polytope, with symmetry group $\mathbf{H} = \langle S_0, \ldots, S_{m-1} \rangle$ and initial vertex w. Define $W := S_0 \cap \cdots \cap S_{m-1}$ and let $v \in W$. Define $R_0 := \{w\}$ as a point-reflexion, $R_j := S_{j+1}$ for $j \geqslant 1$, and let $\mathbf{G} := \langle R_0, \ldots, R_m \rangle$.

The product of the point-reflexions in $a, b \in \mathbb{E}$ is the translation by $2(b - a)$. Thus $\mathbf{G}$ contains all translations by vectors $2(w_k - w_j)$, with $w_j, w_k \in \operatorname{vert} Q$, and so will be discrete only if Q is rational, meaning that its vertices have rational coordinates with respect to some basis.

Effectively, the only choice for v is $v \in \operatorname{aff} \operatorname{vert} Q$ or $v \notin \operatorname{aff} \operatorname{vert} Q$. In the former case, the resulting apeirotope is denoted $P := Q^{\alpha}$; the class of the latter apeirotopes is denoted $\operatorname{apeir} Q$, whose general member is of the form $P \# \{2\}$.

Further comments

With both κ and α, polytopality of the result is not guaranteed, and so must be checked in each individual case. In contrast, if Q is a rational finite regular polytope, then a general member $Q^{\alpha} \# \{2\} \in \text{apeir } Q$ is always polytopal.

There are interesting connexions among these constructions. It is clear from the definition that $\alpha\kappa = \zeta\alpha$.

However, considerably more interesting is a connexion involving Petrie contraction: $\alpha\varpi = \kappa$, as applied to finite polytopes. To see this is easy, since, with v as before the initial vertex of Q with symmetry group $\langle S_0, \ldots, S_{m-1} \rangle$, the effect of $\alpha\varpi$ on the group generators is

$$(S_0, \ldots, S_{m-1}) \mapsto (\{v\}, S_0, \ldots, S_{m-1}) \mapsto (S_0, \{v\}S_1, S_2, \ldots, S_{m-1}),$$

which is just that of κ.