Nearly full rank:
higher dimensions

The gateway

Dimension 5 is the 'gateway' through which we must pass to get at higher dimensions; it is narrow, because there are few possibilities for the symmetry groups of finite regular 4-polytopes in $\mathbb{E}^5$.

The pattern of approach (in all dimensions) is

- pick a possible vertex-figure Q,
- with $\mathbf{H}$ the group of Q, identify those groups $\mathbf{G}$ for which $\mathbf{H} < \mathbf{G}$,
- from the axis of $\mathbf{H}$ in $\mathbf{G}$, obtain the vertex-set V of the corresponding polytope P,
- check whether vert Q is an appropriate subset of V.

In $\mathbb{E}^5$, the group $\mathbf{G}$ must be a subgroup of $\mathbf{A}_5 \times C_2$ or $\mathbf{C}_5$, even (as it turns out) for handed polytopes.

First Gosset class

These polytopes are derived by a mixing operation from a suitable diagram $\mathbf{E}_d$ or $\mathbf{T}_{d+1}$; that is, a group $[3^{r,s,1}]$. The result is

Theorem

For each $r \geqslant 0$ and $s \geqslant 2$ with $(r-1)(s-1) \leqslant 4$ there is a regular polytope (or apeirotope) G_{rs} of nearly full rank with the following properties:

- *it has rank $r+s+1$,*
- *its group $\mathbf{G}_{rs}$ is $[3^{r,s,1}]$,*
- *it has the same vertices as the Gosset polytope r_{s1},*
- *its $(r+2)$-faces are $(r+2)$-cross-polytopes $\{3^r, 4\}$,*
- *its $(s+1)$-cofaces are Petrie contractions $\{3^{s+1}\}^{\varpi}$ of $(s+2)$-simplices.*

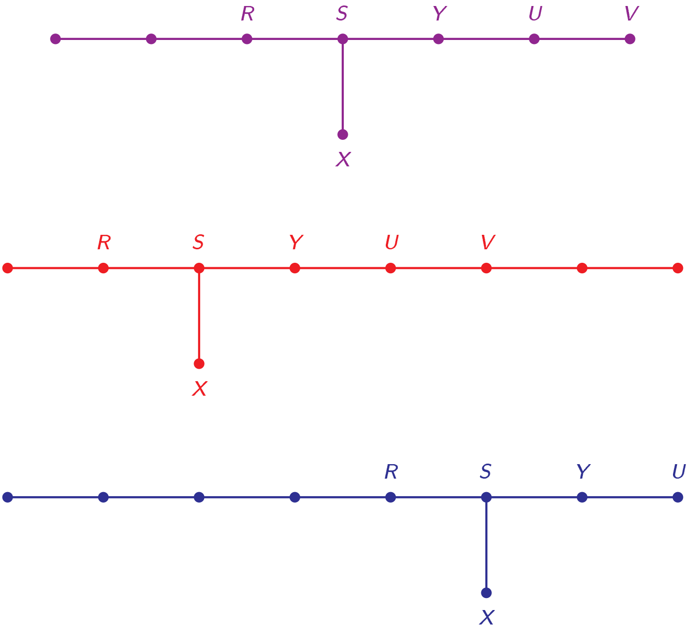
Note that the 3-face of $\{3^{s+1}\}^{\varpi}$ is $\{4, \frac{6}{2,3} \mid 3\}$.

The diagrams

There follow diagrams for the apeirotopal members of the class; those for the polytopes are obtained by deleting nodes from the head or tail of the horizontal part.

For the group generators, the pair X, Y is replaced by the single new generator $T := XY$; as remarked, this is a mixing operation. This, of course, is exactly how Petrie contraction is involved.

An important feature is that X, Y can be recovered from the rest of the symmetry group (S, U are both necessary), which accounts for the symmetry group being preserved.


R
S
Y
U
V
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R
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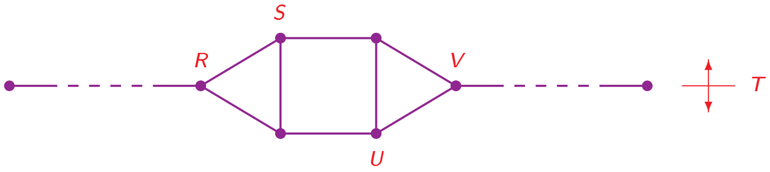
Twisting

Later, we have to twist diagrams with redundant generators. Without redundancy, the only feasible diagrams are of the form



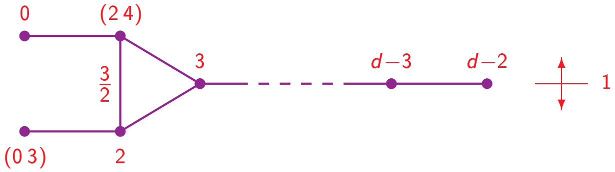
Branches may carry marks, and either vertical branch may be absent. The left and right branches form strings, and either may be absent (including the nodes R or V). The twist is denoted by T.

Example

The group generators of $\{3^{d-1}\}^{\varpi}$ can be taken as the permutations

$$\begin{aligned} R_0 &= (1\ 2), \\ R_1 &= (0\ 1)(2\ 3), \\ R_j &= (j{+}1\ j{+}2), \quad \text{for } j = 2, \ldots, d-2. \end{aligned}$$

However, we can also represent this by a diagram



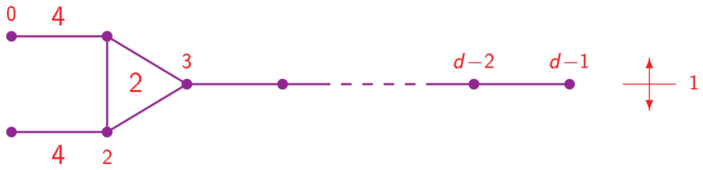
Here, the twist R_1 is inner and proper. However, if we replace R_1 by $-R_1 = -(0\ 1)(2\ 3)$, which removes the mark $\frac{3}{2}$ from the diagram (and places a mark 2 in the triangle, or $\frac{3}{2}$ on one of the other branches), then we have an outer and improper twist.

The polytope now obtained is actually $\{3^{d-1}\}^{\zeta\varpi}$, and so we have doubled the order of the group, as we should expect.

A surprise (perhaps) is that some of the diagrams we twist are far from slight modifications of the usual Coxeter diagrams. Indeed, they may contain quite small subdiagrams which are already those of infinite groups.

Examples

We begin with an infinite family which is already known. We have remarked that the facet of the regular apeirotope $\{4, 3^{d-2}, 4\}^{\kappa}$ is full-dimensional but infinite, and hence of nearly full rank. This facet is $\{4, 3^{d-2}\}^{\kappa}$, and is derivable by an improper outer twist of the diagram



We have labelled the triangle rather than one of its branches; either of its two slanting branches can carry the label $\frac{3}{2}$, but the twist changes which is so labelled. The subdiagram formed by the four leftmost nodes is of the infinite group $[4, 3, 4]$.

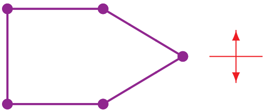
There are further infinite families of diagrams which exhibit the same phenomenon, namely, having a proper subdiagram of an infinite group. We first have

with $q = 3, 4$. In effect, we have two families here, of which the facets of the second form the first. Once again, the corresponding apeirotopes are already known; they are $\{3^{d-1}\}^{\kappa}$ if $q = 3$ or $\{3^{d-2}, 4\}^{\kappa}$ if $q = 4$.

When $d = 4$ (and $q = 3$), we can apply $\boldsymbol{\pi}$ to get the universal

$$\{3,3,3\}^{\boldsymbol{\kappa\pi}} = \{\{6, \tfrac{4}{1,2} \mid 3\}, \{\tfrac{4}{1,2}, 3 : \tfrac{6}{1,3}\}\} \cong \boldsymbol{\{\{}6,4 \mid 3\boldsymbol{\}}, \boldsymbol{\{}4,3\boldsymbol{\}\}},$$

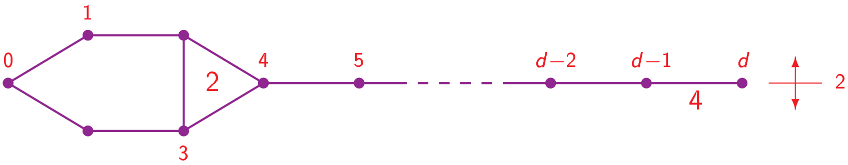
which is obtained by applying a proper outer twist to



Reversing $\boldsymbol{\pi}$ shows that we originally had the universal

$$\{\{6, \tfrac{6}{1,3} \mid 3\}, \{\tfrac{6}{1,3}, 3 : \tfrac{4}{1,2}\}\} \cong \boldsymbol{\{\{}6,6 \mid 3\boldsymbol{\}}, \boldsymbol{\{}6,3 : 4\boldsymbol{\}\}}.$$

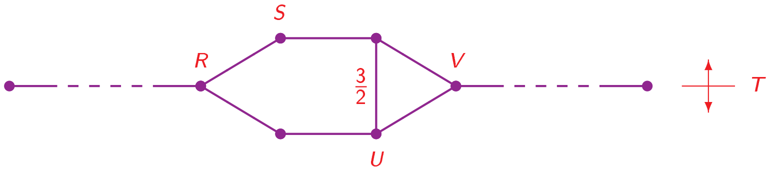
We finally mention



of which the first or last nodes (or both) can be truncated to obtain three other families. Here, the twist is outer and improper.

There are further relatives to these polytopes (we can apply π to the 5-face, and so twist an unmarked hexagonal diagram), but we shall not go into more details.

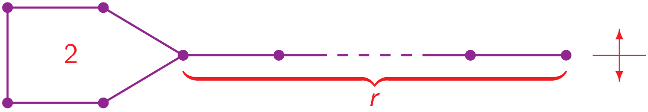
In this spirit, we can also derive the polytopes G_{rs} from diagrams, acted upon by a proper inner twist T. The crucial centre part of the diagram is



We will have $r \geqslant 2$ here, because we have already dealt with the cases $r = 0, 1$ earlier (as two of the set of four families considered together).

In a similar way, G_{13}^{π} can be derived from applying an improper inner twist to a hexagonal diagram with unmarked branches and a central mark 2 (the group is $\mathbf{B}_6$).

The polytopes G_{rs} do not have geometric duals. Nevertheless, we can remove the right branch from the previous diagram (beginning with the node V) and reverse it, to obtain the diagram below. The mark 2 on the circuit indicates as before that one of the branches – but not the vertical one – is to be thought of as marked $\frac{3}{2}$.



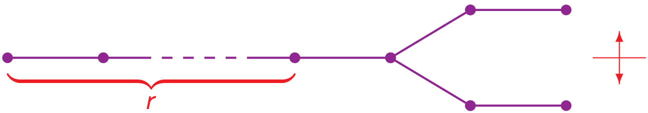
The group specified by the diagram is, again as before, $[3^{r,2,1}]$, so that we must have $r \leqslant 5$, with $r = 5$ the infinite case. We can then – potentially at least – apply an improper twist, to obtain a regular polytope P_n of rank $n = r + 3$.

However, we do not obtain the full family as might be expected. First, $P_4 = \{4, 3^3\}^{\varpi\pi}$, the Petrial of the Petrie contraction of the 5-cube, whose 80 vertices are the mid-points of the edges of the cube. Its facets $\{6, \frac{4}{1,2} \mid 3\}$ (with 30 vertices) lie in central sections of the cube; it follows that P_{n-1} must be a central section of P_n for $n \geqslant 5$, if the latter polytope exists. We immediately conclude – if for no other reason – that we cannot actually have an apeirotope P_8 derived from $[3^{5,2,1}]$.

In fact (to be brief), the situation is even worse: while P_6 does exist, the construction breaks down in rank 7. Calculations show that P_7 would have the 2160 vertices of 2_{41} with the same group $[3^{4,2,1}]$. But the vertex-figure would have the whole symmetry group $[3^{4,1,1}]$ of its vertex-figure, so that that P_7 degenerates. From the group order, it should actually have $192 \cdot 10!/2^6 \cdot 6! = 15120$ vertices; in fact, the vertices collapse in 7s.

Second Gosset class

This family arises from $[3^{r,2,2}]$ with a proper outer twist, namely,



The group is infinite for $r = 2$, so the only cases are $r = 0, 1, 2$. The case $r = 0$ is the universal polytope

$$\{\{3,4\},\{4,\tfrac{4}{1,2} \mid 3\}\} \cong \mathbf{\{\{3,4\},\{4,4 \mid 3\}\}}.$$

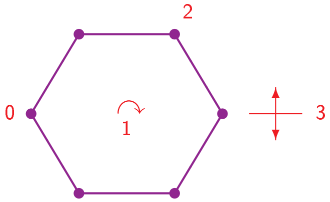
The polytope J_{r+4} (say – $r+4$ is the rank) has as facet the cross-polytope $\{3^{r+1},4\}$, and for $r = 1, 2$ as vertex-figure J_{r+3}; it is then universal with this facet and vertex-figure. The vertex-set is that of r_{22}, and each polytope has a geometric dual.

Petriality π cannot be applied to J_m, but halving η can. In fact, for $m = 4$ we obtain a family

$$\begin{array}{ccccc}
\{4, \frac{4}{1,2}, 3\} & \overset{\delta}{\longleftrightarrow} & \{3, 4, \frac{4}{1,2}\} & & \\
 & & \eta\big\downarrow & & \\
 & & \{4, \frac{4}{1,2}, 4\} & \overset{\delta}{\longleftrightarrow} & \{\frac{4}{1,2}, 4, \frac{4}{1,2}\} \\
 & & & & \eta\big\downarrow \\
 & & & & \{3, \frac{4}{1,2}, 4\}
\end{array}$$

Here, we read $\{4, \frac{4}{1,2}\}$ as $\{4, \frac{4}{1,2} \mid 3\}$ and $\{\frac{4}{1,2}, 4\}$ as $\{\frac{4}{1,2}, 4 \mid 3\}$ (thus $\{\frac{4}{1,2}, 4, \frac{4}{1,2}\} = \{\{\frac{4}{1,2}, 4 \mid 3\}, \{4, \frac{4}{1,2} \mid 3\}\}$). All these polytopes are universal of their types; the last has no geometric dual.

Now π is applicable to $\{4, \frac{4}{1,2}, 4\}$ and $\{\frac{4}{1,2}, 4, \frac{4}{1,2}\}$, with rather messy symbols. From the application of η, the group of the polytope $\{4, \frac{4}{1,2}, 4\}$ and its Petrial contains two twists; to incorporate these into a diagram we need redundant generators. Thus we have



Applying π replaces R_1 by the (twist) reflexion of the diagram in its vertical line of symmetry.

For $m = 5$, a different idea (coming from a simplex dissection formula) leads to a sequence

$$\{3,3,4,\tfrac{4}{1,2}\},\quad \{3,4,\tfrac{4}{1,2},4\},\quad \{4,\tfrac{4}{1,2},4,\tfrac{4}{1,2}\},$$

$$\{\tfrac{4}{1,2},4,\tfrac{4}{1,2},3\},\quad \{4,\tfrac{4}{1,2},3,3\};$$

it begins with J_5 and J_5^η, and reverses under geometric duality δ.

For $m = 6$, we similarly have

$$\{3,3,3,4,\tfrac{4}{1,2}\},\quad \{3,3,4,\tfrac{4}{1,2},4\},\quad \{3,4,\tfrac{4}{1,2},4,\tfrac{4}{1,2}\},$$

$$\{4,\tfrac{4}{1,2},4,\tfrac{4}{1,2},3\},\quad \{\tfrac{4}{1,2},4,\tfrac{4}{1,2},3,3\},\quad \{4,\tfrac{4}{1,2},3,3,3\};$$

with the same behaviour as for $m = 5$.

There are now two more opportunities to apply halving η; in what follows, $\{4, 3, \frac{4}{1,2}\} = \{4, 3, \frac{4}{1,2} \mid 3\}$ is a toroid. Note the occurrence of the 24-cell $\{3, 4, 3\}$ as the 4-face of the latter.

$$\{4, \tfrac{4}{1,2}, 4, \tfrac{4}{1,2}\}^{\eta} = \{4, 3, \tfrac{4}{1,2}, 4\},$$
$$\{3, 4, \tfrac{4}{1,2}, 4, \tfrac{4}{1,2}\}^{\eta} = \{3, 4, 3, \tfrac{4}{1,2}, 4\}.$$

There are no further applications of π; however, there are of ζ or κ, as appropriate, but again the results are of not of much interest, except in so far as they complete classifications.

Rotational symmetry group

As it happens, in $\mathbb{E}^d$ for $d \geqslant 5$ there is just one handed polytope of nearly full rank. This is

$$\{\{4, \tfrac{5}{1,2} : \tfrac{5}{1,2}\}, \{\tfrac{5}{1,2}, 3 : \tfrac{5}{1,2}\}\} \cong \boldsymbol{\{\{4, 5 : 5\}, \{5, 3 : 5\}\}},$$

the universal regular 4-polytope of its type.

For handedness, the vertex-figure must have mirror vector $(2, 2, 2)$; the initial considerations then eliminate all but those to which a hyperplane reflexion (acting as the Petrie operation $\boldsymbol{\pi}$ or something similar) can be adjoined, and finally we are reduced to $\{\tfrac{5}{1,2}, 3 : \tfrac{5}{1,2}\}$.

As might be suspected, this polytope is self-Petrie.

Applying ζ to $\{\{4, \frac{5}{1,2} : \frac{5}{1,2}\}, \{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}\}$ leads to a non-handed polytope.

There are no handed apeirotopes of nearly full rank in $\mathbb{E}^5$, nor any such polytopes in $\mathbb{E}^d$ for $d \geqslant 6$. The only possible vertex-figure for such a polytope of rank 5 is $\{\{4, \frac{5}{1,2} : \frac{5}{1,2}\}, \{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}\}$ itself. It is tempting to look for a polytope with the vertices of 2_{21} (for example), but all possible constructions break down.

We conclude that the occurrence of handed regular polytopes (or apeirotopes) of nearly full rank is a low-dimensional phenomenon. And, of course, the symmetry group of a regular polytope of full rank contains hyperplane reflexions, and so such a polytope cannot be handed.

A remarkable polyhedron

The facet $\{4,5:5\}$ of $\{\{4,5:5\},\{5,3:5\}\}$ is of considerable interest in its own right. Both the facetting operation φ_2 and the combined operation of halving followed by the Petrie operation $\eta\pi$ result in regular polyhedra of type $\{4,5\}$ with the same group. They are actually isomorphic copies of $\{4,5:5\}$.

The polyhedron $\{4,5:5\}$ has three non-trivial diagonal classes, with layer vector $(1,5,5,5)$. It may be seen that φ_2 interchanges the second and third diagonal classes, while $\eta\pi$ permutes them cyclically. Thus, on a general cosine vector, these two operations have the following effect:

$$\varphi_2\colon (\gamma_0,\ldots,\gamma_3) \mapsto (\gamma_0,\gamma_1,\gamma_3,\gamma_2),$$
$$\eta\pi\colon (\gamma_0,\ldots,\gamma_3) \mapsto (\gamma_0,\gamma_2,\gamma_3,\gamma_1).$$

As a consequence, one general realization of $\{4, 5 : 5\}$ will give rise to five others with the same vertices. The pure realizations are all 5-dimensional, and the cosine matrix can be written in a nice symmetric form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ 1 & \frac{1}{5} & \frac{1}{5} & -\frac{3}{5} \end{bmatrix}.$$

The handed polytope $\{\{4, \frac{5}{1,2} : \frac{5}{1,2}\}, \{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}\}$ has facet $\{4, \frac{5}{1,2} : \frac{5}{1,2}\} =: P_2$ in the implied list, as can be checked directly from the geometry. It is rigid, since it is the only realization with planar faces $\{4\}$.