4-dimensional 4-apeirotopes

Preliminary remarks

A number of topics introduced in this part will carry over to higher dimensions. Conversely, the applications of some constructions (particularly) which work in all dimensions will not be treated here. Thus we shall largely concentrated on those aspects which are peculiar to 4-dimensional space.

In fact, as we have seen already, $\mathbb{E}^4$ admits a rich variety of regular polyhedra; there is similarly a rich variety of 4-dimensional regular apeirotopes of rank 4, since these polyhedra constitute their vertex-figures.

Mirror vectors provided the way in for the polyhedra. In contrast, for the apeirotopes, the groups themselves form the necessary framework for the classification.

Nearest-point criterion

The following obvious result is frequently useful. If X is a finite set of points in an ambient space $\mathbb{A}$ (a sphere or euclidean space) and $v \notin X$ is a further point, denote by $\alpha(v, X)$ the smallest angle subtended at v in $\mathbb{A}$ by some pair of points of X.

Theorem

Let P be a regular polytope (apeirotope) with initial vertex v, and let X be the set of vertices of P (other than v) nearest to v. Then $\alpha(v, X) > \pi/3$ ($\alpha(v, X) \geqslant \pi/3$, respectively).

Proof.

If the given condition failed to hold, then two points of X would be closer to each other than they are to v, a contradiction to the definition of X and the regularity of P. (Note that the angle of a spherical equilateral triangle is greater than $\pi/3$.) $\square$

Blends

Naturally, among the 4-dimensional regular apeirotopes of nearly full rank will be the blends of the eight 3-dimensional pure regular 4-apeirotopes with the digon $\{2\}$ or apeirogon $\{\infty\}$.

Among the blends with $\{2\}$ are the general members of apeir Q, with Q a 3-dimensional crystallographic polytope (one of $\{3,3\}$, $\{3,4\}$, $\{4,3\}$ and their Petrials).

Blends with $\{2\}$ are the only regular 4-apeirotopes of nearly full rank which can have 3-dimensional vertex-figures. Thus we can confine our attention to the apeirotopes whose vertex-figures are 4-dimensional regular polyhedra Q, which we have previously discussed. Furthermore, we need not consider the free abelian apeirotopes Q^{α} for such rational Q.

Blended vertex-figures

An apeirotope here is obtained by applying κ to a crystallographic regular 4-polytope Q. These 4-polytopes are paired up by ζ.

Each classical crystallographic 4-polytope Q is the facet of a honeycomb P; thus Q^κ will correspondingly be the facet of P^κ.

Since the 2-faces of an apeirotope Q^κ are even (or apeirogons), Petriality π can always be applied; note that $\pi\kappa = \kappa\pi$ in case Q^π exists.

Note that, if Q is a classical regular polytope with 2-faces $\{p\}$, then the 2-faces of $Q^{\zeta\kappa}$ are helices $\{\frac{p}{0,1}\}$.

Remark

Similar results hold for applications of κ to regular polytopes of any full rank.

It may help here to indicate how the mirror vectors change under these operations (note that ζ must be applied before κ or $\kappa\pi$).

$$\begin{tikzcd}
(3,3,3,3) \arrow[r, leftrightarrow, "\kappa"] \arrow[d, leftrightarrow, "\zeta"'] & (3,1,3,3) \arrow[r, leftrightarrow, "\pi"] \arrow[d, leftrightarrow, "\zeta"'] & (3,2,3,3) \arrow[d, leftrightarrow, "\zeta"'] \\
(1,3,3,3) \arrow[r, leftrightarrow, "\kappa"] & (1,1,3,3) \arrow[r, leftrightarrow, "\pi"] & (1,2,3,3)
\end{tikzcd}$$

At top left are the classical regular 4-polytopes; moreover, the facets at top middle are Petrie-Coxeter apeirohedra (in two cases, those at top right are as well).

Possible groups

From now on, we can assume that the vertex-figure of our discrete regular 4-apeirotope P is a 4-dimensional regular polyhedron Q. Moreover, if we leave until last the discussion of handed vertex-figures, the symmetry group $\mathbf{G}$ of P will be a hyperplane reflexion group, possibly acted on by outer automorphisms.

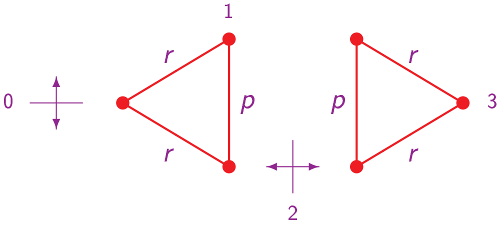
As a consequence, $\mathbf{G}$ will be a subgroup of one of

- $(\mathbf{V}_3 \times \mathbf{V}_3) \rtimes \boldsymbol{C}_2$, the product of two groups of the triangular plane tiling acted on by an involutory automorphism;
- $\mathbf{P}_5 \rtimes \boldsymbol{D}_5$, the unmarked pentagon acted on by the dihedral group;
- $\mathbf{U}_5 = [3, 3, 4, 3]$.

The last subsumes the group $\mathbf{R}_5 = [4, 3, 3, 4]$ of the cubic tiling.

The case $(\mathbf{V}_3 \times \mathbf{V}_3) \rtimes \boldsymbol{C}_2$

The basic picture here is the following diagram $\mathcal{A}_2(p, r)$ with twists.



The decomposition $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$ suggests the use of complex coordinates. Our starting point is the case $(p, r) = (\frac{3}{2}, 6)$, but note that many of the same arguments apply to the other euclidean triangular diagrams, namely, the cases $(p, r) = (3, 3),\ (2, 4)$.

A useful operation

With $z = (x, y)$, and writing $\omega := \exp(2i\pi/3) = \frac{1}{2}(-1 + \sqrt{3})$ and $\varphi := -\omega^2 = \exp(i\pi/3) = \frac{1}{2}(1 + \sqrt{3})$, the symmetry group of $\{4, \frac{4}{1,2} \mid 6\}^{\boldsymbol{\alpha}}$ has generators

$$S_j : z \mapsto \begin{cases} (1 - x, 1 - y), & \text{if } j = 0, \\ (\varphi\overline{x}, y), & \text{if } j = 1, \\ (y, x), & \text{if } j = 2, \\ (x, \overline{y}), & \text{if } j = 3. \end{cases}$$

The initial vertex is the origin, and that of the vertex-figure is $(1, 1)$.

We now define

$$T := (S_2 S_3)^2 : z \mapsto (\overline{x}, \overline{y}),$$

set $R_0 := S_0 T$ and $R_j := S_j$ for $j = 1, 2, 3$.

This yields an apeirotope $\{\{\frac{6}{2,3},4:\frac{4}{1,2}\},\{4,\frac{4}{1,2}\mid 6\}\}$, to which we can successively apply halving $\boldsymbol{\eta}$. However, observe that

$$z(R_1R_2R_1R_0) = (1+\omega y, 1+\omega x),$$

so that $z(R_1R_2R_1R_0)^3 = (y,x) = zR_2$. Consequently, when we apply

$$\boldsymbol{\eta}\colon (R_0,\ldots,R_3) \mapsto (R_0, R_1R_2R_1, R_3, R_2)$$

the first time, the new group violates the intersection property, which means that we obtain a non-polytope.

However, a double application of $\boldsymbol{\eta}$ effectively leads us to the diagram $\mathcal{A}_2(3,3)$; then we have

$$\{\{\tfrac{6}{1,3},4:\tfrac{4}{1,2}\},\{4,\tfrac{4}{1,2}\mid 3\}\}^{\boldsymbol{\delta}} = \{\{4,\tfrac{4}{1,2}\mid 3\},\{\tfrac{4}{1,2},6:\tfrac{4}{1,2}\}\},$$
$$\{\{\tfrac{6}{1,3},4:\tfrac{4}{1,2}\},\{4,\tfrac{4}{1,2}\mid 3\}\}^{\boldsymbol{\eta}} = \{\{6,\tfrac{4}{1,2}\mid \tfrac{6}{1,2}\},\{\tfrac{4}{1,2},4\mid 3\}\}.$$

The notation for the last is deficient.

We shall not go further into this class, except to note that, although we can apply various other operations such as π or κ, in many cases the results degenerate, or – even if they do exist – they are relatively uninteresting. In any event, usually we cannot determine the combinatorial type with any ease.

The case $\mathbf{P}_5 \rtimes D_5$

In fact, here we only have the group $\mathbf{P}_5 \rtimes C_2$. The apeirotopes in this family take their start from $\{3,3,3\}^{\kappa}$, which we discussed earlier.

It is worth noting that the full group $\mathbf{P}_5 \rtimes D_5$ does occur as that of the facet $\{\frac{5}{1,2}, \frac{4}{1,2} \mid \frac{6}{1,3}\}^{\pi}$ of $\{3,3,3\}^{\zeta\kappa\pi}$; here,

$$\{\tfrac{5}{1,2}, \tfrac{4}{1,2} \mid \tfrac{6}{1,3}\} \cong \boldsymbol{\{}5,4 \mid 6\boldsymbol{\}}$$

is universal. Further,

$$\{3,3,3\}^{\kappa\pi} = \{\{6, \tfrac{4}{1,2} \mid 3\}, \{\tfrac{4}{1,2}, 3 : \tfrac{6}{2,3}\}\} \cong \boldsymbol{\{\{}6,4 \mid 3\boldsymbol{\}}, \boldsymbol{\{}4,3\boldsymbol{\}\}}$$

is also universal.

The last of these apeirotopes has a geometric dual

$$\{\{3,4\},\{4,\tfrac{6}{1,3}\mid 3\}\} \cong \pmb{\{\{3,4\},\{4,6\mid 3\}\}},$$

to which κ can be applied. This yields the universal

$$\begin{aligned}\{\{3,4\},\{4,\tfrac{6}{1,3}\mid 3\}\}^{\kappa} &= \{\{6,\tfrac{4}{1,2}\mid 4\},\{\tfrac{4}{1,2},\tfrac{6}{1,3}:\tfrac{5}{1,2}\}\}\\ &\cong \pmb{\{\{6,4\mid 4\},\{4,6:5\}\}}.\end{aligned}$$

Observe that the facet is a Petrie-Coxeter sponge.

We can now apply π to the last, to obtain

$$\{\{6,\tfrac{4}{1,2}\mid 4\},\{\tfrac{4}{1,2},\tfrac{6}{1,3}:\tfrac{5}{1,2}\}\}^{\pi} = \{\{6,\tfrac{5}{1,2}\mid 4\},\{\tfrac{5}{1,2},\tfrac{6}{1,3}:\tfrac{4}{1,2}\}\}.$$

The facet and vertex-figure are universal of their kinds; whether the whole apeirotope is universal is unclear.

There are also applications of η, after which other operations will apply.

$$\{\{3,4\},\{4,\tfrac{6}{1,3} \mid 3\}\}^{\eta} = \{\{4,\tfrac{6}{1,3} \mid 3\},\{\tfrac{6}{1,3},6:\tfrac{6}{2,3}\}\},$$
$$\{\{4,\tfrac{6}{1,3} \mid 3\},\{\tfrac{6}{1,3},6:\tfrac{6}{2,3}\}\}^{\pi} = \{\{4,\tfrac{6}{2,3} \mid 3\},\{\tfrac{6}{2,3},6:\tfrac{6}{1,3}\}\},$$
$$\{\{4,\tfrac{6}{1,3} \mid 3\},\{\tfrac{6}{1,3},6:\tfrac{6}{2,3}\}\}^{\delta} = \{\{\tfrac{6}{1,3},6:\tfrac{6}{2,3}\},\{6,\tfrac{4}{1,2} \mid 3\}\}.$$

The vertex-figures here are not universal by any means.

These are jumping-off points for applying κ, π.

$$
\begin{array}{ccc}
\{\{\frac{6}{1,3}, 6 : \frac{6}{2,3}\}, \{6, \frac{4}{1,2} \mid 3\}\} & \overset{\kappa}{\longleftrightarrow} & \{\{\frac{3}{0,1}, \frac{6}{2,3} : \frac{6}{0,1}\}, \{\frac{6}{2,3}, \frac{4}{1,2} : \frac{5}{1,2}\}\} \\
\pi \updownarrow & & \pi \updownarrow \\
\{\{\frac{6}{1,3}, \frac{10}{1,3} : \frac{6}{2,3}\}, \{\frac{10}{1,3}, \frac{4}{1,2} : 6, 3\} & \overset{\kappa}{\longleftrightarrow} & \{\{\frac{3}{0,1}, \frac{5}{1,2} : \frac{6}{0,1}\}, \{\frac{5}{1,2}, \frac{4}{1,2} : \frac{6}{2,3}\}\}
\end{array}
$$

There is an analogous set derived from $\{\{\frac{6}{1,3}, 4 : \frac{6}{1,3}\}, \{4, \frac{6}{2,3} \mid 3\}\}$, which is closely related to $\{4, \frac{6}{2,3} \mid 3\}^{\alpha}$ (the point reflexion is replaced by one in a line). However, we shall not list these, in part because their exact combinatorial structures are still not determined.

The case $[3,3,4,3]$

In the final part, we shall deal with applications of the various operations which result in infinite families related to the simplices, cross-polytopes and cubes. Here, we briefly look at the 24-cell and its related tessellations.

First consider $\{3,4,3\}^{\boldsymbol{\kappa}}$. The general discussion of how $\boldsymbol{\kappa}$ works shows that its facet is the apeirohedron

$$\{3,4\}^{\boldsymbol{\kappa}} = \{6, \tfrac{4}{1,2} \mid 4\};$$

this is one of the Petrie-Coxeter sponges. Similarly, the new vertex-figure is

$$\{4,3\} \diamond \{2\} = \{4,3\}^{\boldsymbol{\zeta}} \# \{2\} = \{\tfrac{4}{1,2}, 3 : 3\} \# \{2\} = \{\tfrac{4}{1,2}, 3 : \tfrac{6}{2,3}\}.$$

The last fine Schläfli symbol does specify the geometry of the vertex-figure. Indeed, we now have a fine Schläfli symbol for $\{3,4,3\}^{\kappa}$ itself, namely,

$$\{3,4,3\}^{\kappa} = \{\{6, \tfrac{4}{1,2} \mid 4\}, \{\tfrac{4}{1,2}, 3 : \tfrac{6}{2,3}\}\}.$$

This is actually rigid, because the facets are rigid, and the planar 2-face $\{6\}$ fixes the relative sizes of the components $\{\frac{4}{1,2}, 3 : 3\}$ and $\{2\}$ of the blended vertex-figure.

We shall not say more about the family resulting from this. Moreover, the families derived by κ from $\{3,3,4\}$ and $\{4,3,3\}$ will be discussed in the next part; by and large, the extra apeirotopes obtained by applying π are not of much interest, except from the viewpoint of a complete enumeration.

We leave $\mathbf{U}_5$, though, with an interesting subfamily:

$$\begin{tikzcd}
\{\{\frac{6}{1,3}, 4 : \frac{6}{1,3}\}, \{4, \frac{4}{1,2} \mid 4\}\} \arrow[r, "\eta"] \arrow[d, leftrightarrow, "\tau"'] & \{\{\frac{2}{0,1}, \frac{4}{1,2} : \frac{4}{1,2}\}, \{\frac{4}{1,2}, 4 : \frac{4}{1,2}\}\} \arrow[d, leftrightarrow, "\tau"'] \\
\{\{3,4\}, \{4, \frac{4}{1,2} \mid 4\}\} \arrow[r, "\eta"] & \{\{4, \frac{4}{1,2} \mid 4\}, \{\frac{4}{1,2}, 4 : \frac{4}{1,2}\}\} \arrow[d, leftrightarrow, "\delta"'] \\
\{\{4, \frac{4}{1,2}\}, \{\frac{4}{1,2}, 4 : \frac{4}{1,2}\}\} & \{\{\frac{4}{1,2}, 4 : \frac{4}{1,2}\}, \{4, \frac{4}{1,2} \mid 4\}\} \arrow[l, "\eta"']
\end{tikzcd}$$

Here, τ replaces a hyperplane reflexion by one in a line. Recall that $\{\frac{2}{0,1}\}$ is the zigzag apeirogon; at bottom left, the facet is isomorphic to $\{4,4\}$.

Handed vertex-figures

Most of the putative handed vertex-figures are excluded by the nearest-point criterion, particularly those whose left and right quaternion groups constituting their symmetry groups are different. Of course, acceptable polyhedra must be 'crystallographic', in a sense which we shall not make too precise.

And, of those potential vertex-figures which survive this winnowing, many only permit the free abelian apeirotope construction α. Unfortunately, these include the pentagonal polyhedra, such as the hemi-dodecahedron $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$, the dodecahedron $\{\frac{5}{1,2}, 3 : \frac{10}{1,3}\}$ and its Petrial, and $\{\frac{10}{1,3}, 3 : \frac{10}{1,3}\}$.

Remark

In fact, some do lead to pre-apeirotopes, which become polytopal on blending with $\{2\}$.

We finally conclude that the only polyhedra of this kind which can be vertex-figures in a non-trivial way are

$$\{2,3:6\} \bowtie \{6,3:2\} = \{\tfrac{6}{12}, 3 : \tfrac{6}{1,2}\},$$
$$\{2,4:4\} \bowtie \{4,4:2\} = \{\tfrac{8}{1,3}, 4 : \tfrac{8}{1,3}\},$$
$$\{2,6:3\} \bowtie \{6,3:2\} = \{\tfrac{12}{1,5}, 3 : \tfrac{12}{1,5}\}.$$

Defining $\varphi := \exp(2i\pi/q)$, with $q = 3, 4, 6$, respectively, the symmetry groups of the vertex-figures are generated by

$$R_1 : z \mapsto (y, x),$$
$$R_2 : z \mapsto (\overline{x}, \varphi\overline{y}),$$
$$R_3 : z \mapsto (\overline{x}, \overline{y}).$$

The different choices for R_0 are

$$z \mapsto (1 - x, \mp y) \text{ or } (1 - \overline{x}, \mp y).$$

The mirror R_0 then has dimension $0, 2, 1, 3$, respectively.

In every case, $\langle R_1 R_0 R_1, R_2, R_3 \rangle$ is the group of the planar tiling $\{p, q\}$ (for the appropriate p). This then excludes mirror dimensions $1, 3$ in case $q = 6$, because an odd edge-circuit would take the vertex-figure back into its enantiomorphic copy. All other choices give rise to genuine regular apeirotopes.

Remark

Few of the regular polyhedra Q in mirror class $(2, 2, 2)$ are rational, which is necessary that Q^{α} be definable.