Realizations

Basic notions

Let G be a countable group acting transitively on a set $\mathcal{V}$. Thus $\mathcal{V}$ itself is countable, and so we can always think of $\mathcal{V}$ as ordered in some way. Pick $e \in \mathcal{V}$ any element, and let H be the stabilizer of e in $\mathcal{V}$. Thus we may identify $\mathcal{V}$ with the family of (right) cosets Hx of H in G, and write x for the corresponding element of $\mathcal{V}$. However, it is helpful to retain $\mathcal{V}$ as a separate entity.

Remark

In particular, we can identify e with the identity of G.

A realization of $(\mathcal{V}, \boldsymbol{G})$ is a mapping $\Psi\colon \mathcal{V} \times \boldsymbol{G} \to \mathbb{E} \times \mathbf{M}$, with $\mathbf{M}$ the group of isometries on the euclidean space $\mathbb{E}$, such that

$$(\boldsymbol{x}\boldsymbol{g})\Psi = (\boldsymbol{x}\Psi)(\boldsymbol{g}\Psi)$$

for all $\boldsymbol{x} \in \mathcal{V}$ and $\boldsymbol{g} \in \boldsymbol{G}$. In other words, Ψ is compatible with the group action; in particular, Ψ induces a homomorphism on $\boldsymbol{G}$.

We write $\mathbf{G} := \boldsymbol{G}\Psi$ and $V := \mathcal{V}\Psi$. Thus $\mathbf{G}$ is a group of isometries of $\mathbb{E}$ which acts transitively on V.

Remark
Such a geometric situation is often a starting point, with $(V, \mathbf{G})$ playing the rôle of $(\mathcal{V}, \boldsymbol{G})$.

The set of points $\operatorname{axis} \mathbf{H}$ of $\mathbb{E}$ fixed by $\mathbf{H} := \boldsymbol{H}\Psi$ is its axis; the image $v := \boldsymbol{e}\Psi$ is the initial point of the realization.

Quotients

If the realization Ψ is one-to-one on $\mathcal{V}$ and an isomorphism on $\boldsymbol{G}$, then we call it faithful.

Remark

A more subtle concept of faithfulness will be introduced for realizations of polytopes.

Remark

Observe that both conditions for faithfulness are important.

We say that another realization Ω of $(\mathcal{V}, \boldsymbol{G})$ is a quotient of Ψ, if there is an affine mapping $\Phi\colon \mathbb{E} \to \mathbb{E}'$, with $\mathbb{E}'$ another euclidean space, such that $\Omega = \Psi\Phi$.

Remark

Of course, Φ must take one group of isometries $\boldsymbol{G}\Psi$ into another such group $\boldsymbol{G}\Omega$.

There are two basic ways in which quotients can arise.

First, through a homomorphism Φ on $\boldsymbol{G}$, with kernel $\boldsymbol{N}$, say. Then $\boldsymbol{G}$ and $\boldsymbol{H}$ are replaced by $\boldsymbol{G}/\boldsymbol{N}$ and $\boldsymbol{H}/(\boldsymbol{N} \cap \boldsymbol{H})$, respectively.

Remark

Observe that, if $\boldsymbol{N} \leqslant \boldsymbol{H}$, then Φ is one-to-one on $\mathcal{V}$.

Second, $\boldsymbol{G}$ remains the same, but $\boldsymbol{H}$ is replaced by a subgroup $\boldsymbol{H}'$ with $\boldsymbol{H} < \boldsymbol{H}' \leqslant \boldsymbol{G}$.

Remark

In general, therefore, a quotient map Φ will combine elements of both forms, and so will induce identifications on both $\mathcal{V}\Psi$ and $\boldsymbol{G}\Psi$.

Diagonal classes

A diagonal in $\mathcal{V}$ is a pair $\{\boldsymbol{x}, \boldsymbol{y}\}$ of (distinct) elements of $\mathcal{V}$ (for the moment unordered). A diagonal class consists of the family of diagonals equivalent under $\boldsymbol{G}$ to a given one. Since $\mathcal{V}$ is countable, the diagonal classes can be ordered in some fixed way.

If Ψ is a realization, let $\{\boldsymbol{x}, \boldsymbol{y}\}$ represent the kth diagonal class, and write

$$\delta_k = \delta_k(\Psi) := \|\boldsymbol{x}\Psi - \boldsymbol{y}\Psi\|^2.$$

Then $\Delta = \Delta(\Psi) := (\delta_1, \delta_2, \ldots)$ is called the diagonal vector of Ψ.

Theorem

The diagonal vector Δ determines the realization Ψ up to congruence.

Realization cone

It is natural to identify two realizations Ψ and Ω if the corresponding images $\mathcal{V}\Psi$ and $\mathcal{V}\Omega$ are congruent. We now use the notation $\mathcal{V}$ to mean the family of congruence classes of these realizations.

We scale a realization Ψ by $\lambda \in \mathbb{R}$ by $\boldsymbol{x}(\lambda\Psi) := \lambda(\boldsymbol{x}\Psi)$ for each $\boldsymbol{x} \in \mathcal{V}$. We blend realizations Ψ and Ω to form $\Psi \# \Omega$, which is defined by $\boldsymbol{x}(\Psi \# \Omega) := (\boldsymbol{x}\Psi, \boldsymbol{x}\Omega)$ in the product space.

Theorem

If Ψ and Ω are two realizations, then their diagonal classes satisfy

$$\Delta(\lambda\Psi) = \lambda^2 \Delta(\Psi),$$
$$\Delta(\Psi \# \Omega) = \Delta(\Psi) + \Delta(\Omega).$$

Thus $\mathcal{V}$ has the structure of a convex cone.

Polygons

A general regular polygon will be a blend of planar ones, possibly also including the (line) segment $\{2\}$ or the (linear) apeirogon $\{\infty\}$. A planar polygon will be of the form $\{p\}$, with $p = \frac{r}{s} > 2$ generally a fraction, always assumed to satisfy $(r, s) = 1$ and $1 \leqslant s < \frac{1}{2}r$.

The notation for a general regular polygon is then

$$\{p\} = \left\{\frac{r}{s_1, \ldots, s_k}\right\} = \left\{\frac{r}{s_1}\right\} \# \cdots \# \left\{\frac{r}{s_k}\right\},$$

where $0 \leqslant s_1 < \cdots < s_k \leqslant \frac{1}{2}r$ with $(r, s_1, \ldots, s_k) = 1$; the case $s_1 = 0$ gives $\{\infty\}$, while $s_k = \frac{1}{2}r$ gives $\{2\}$. We call the entry p a generalized fraction.

Remark

We exclude here apeirogons with irrational turns.

Example

An r-helix is an apeirogon $\{\frac{r}{0,1}\}$, usually with integer r.

Products

The product of representations of groups only makes sense when they are orthogonal. Thus we now suppose that $\boldsymbol{G}$ is finite, and that all images of $\boldsymbol{G}$ under realizations are orthogonal groups. Then the (tensor) product of realizations Ψ and Ω of $\mathcal{V}$ is $\Psi \otimes \Omega$, given by

$$\boldsymbol{x}(\Psi \otimes \Omega) := (\boldsymbol{x}\Psi) \otimes (\boldsymbol{x}\Omega)$$

for $\boldsymbol{x} \in \mathcal{V}$.

Since we identify congruent realizations, we have

Theorem

Products of realizations are associative and commutative.

It is clear that the product interacts with blends and scaling by

Theorem

If Φ, Ψ, Ω are realizations and $\lambda \in \mathbb{R}$, then

$$\Phi \otimes (\Psi \# \Omega) = (\Phi \otimes \Psi) \# (\Phi \otimes \Omega),$$
$$(\lambda\Phi) \otimes \Psi = \lambda(\Phi \otimes \Psi).$$

While we can calculate the diagonal vector $\Delta(\Psi \otimes \Omega)$, it is far better to work with inner products. Indeed, these obviously suffice, in view of

$$\|\boldsymbol{x}\Psi - \boldsymbol{y}\Psi\|^2 = \|\boldsymbol{x}\Psi\|^2 + \|\boldsymbol{y}\Psi\|^2 - 2\langle \boldsymbol{x}\Psi, \boldsymbol{y}\Psi\rangle,$$

with $\|\boldsymbol{x}\Psi\| = \|\boldsymbol{y}\Psi\|$ the radius of the sphere containing the realization.

Cosine vectors

Up to scaling, we can assume all realizations to be normalized, with points lying on the appropriate unit sphere. Instead of the diagonal vector, we have the cosine vector $\Gamma = \Gamma(\Psi) = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_r)$, given by

$$\gamma_k := \langle \mathbf{x}\Psi, \mathbf{y}\Psi \rangle,$$

where, as before, $\{\mathbf{x}, \mathbf{y}\}$ represents the k-th diagonal class. We now adjoin $\gamma_0 := 1$ to represent the degenerate diagonals $\{\mathbf{x}, \mathbf{x}\}$.

If $a = (\alpha_0, \dots, \alpha_r)$ and $b = (\beta_0, \dots, \beta_r)$, we shall write simply $ab := (\alpha_0\beta_0, \dots, \alpha_r\beta_r)$.

Theorem

The product $\Psi \otimes \Omega$ has cosine vector $\Gamma(\Psi \otimes \Omega) = \Gamma(\Psi)\Gamma(\Omega)$.

There is a multiplicative unit or henagon $\{1\}$, which consists of the single point $1 \in \mathbb{R}$. As the trivial realization, its cosine vector will be $\Gamma_0 = (1^{r+1})$, where α^k in such expressions will stand for a sequence $\alpha, \ldots, \alpha$ of length k.

Remark

To preserve the normalization, we must take blends of the kind $\lambda\Phi \# \mu\Psi$, where $\lambda^2 + \mu^2 = 1$. In terms of cosine vectors, this reduces to taking convex combinations, which ensure that the leading term (corresponding to the trivial diagonal class) remains 1.

Layer vectors

If the jth diagonal class from the initial vertex has ℓ_j members, then

$$\Lambda := (\ell_0, \ell_1, \ldots, \ell_r)$$

is the layer vector. As usual, $\ell_0 = 1$ gives the trivial diagonal. Thus, if a realization is centred, meaning that the centroid of its vertices is the origin o, then its cosine vector Γ must satisfy the layer equation

$$\langle \Lambda, \Gamma \rangle = 0.$$

More generally, a cosine vector Γ will satisfy the layer inequality $\langle \Lambda, \Gamma \rangle \geqslant 0$. Write $|\Lambda| := \ell_0 + \ell_1 + \cdots + \ell_r$ for the total number of points. If $\Gamma = \alpha_0 \Gamma_0 + \alpha_1 \Gamma_1$, a convex combination, with Γ_1 corresponding to the centred component, then $\alpha_0 = \langle \Lambda, \Gamma \rangle / |\Lambda|$.

Wythoff's construction

Let $\mathbf{G}$ be a discrete group acting on a euclidean space $\mathbb{E}$, which is generated by reflexions $R_0, \ldots, R_{m-1}$, that is, involutory isometries; we may take the generating set to be minimal. We call $\mathbf{G}$ connected if its generators cannot be partitioned into two subsets whose elements mutually commute.

In its most general terms, a Wythoff construction is of the following form. As usual, we identify a reflexion R with its mirror of fixed points $\{x \in \mathbb{E} \mid xR = x\}$. If $\mathsf{K} \subseteq \mathsf{M} := \{0, \ldots, m-1\}$, then we write $W_\mathsf{K} := \bigcap\{R_j \mid j \notin \mathsf{K}\}$, which we call a Wythoff space. We pick a (general) point $v \in W_\mathsf{K}$. The images $V := v\mathbf{G}$ form the vertex-set of the realization. Initial edges are of the form $\{v, vR_j\}$ with $j \in \mathsf{K}$, and their images under subgroups of $\mathbf{G}$ then fit together to form polygonal 2-faces, polyhedral 3-faces, and so on (we do not want to be too precise at this stage).

Regular polytopes

In this context, we have a string C-group $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1} \rangle$, as the automorphism group of an abstract regular polytope $\mathcal{P}$, acting on the distinguished subgroup $\boldsymbol{H} := \boldsymbol{G}_0$. Thus $\mathbf{G} = \boldsymbol{G}\Phi$, say, is a representation of $\boldsymbol{G}$. Then the Wythoff space here is the axis $W := W_0$ of $\mathbf{G}_0 = \langle R_1, \ldots, R_{m-1} \rangle = \boldsymbol{G}_0\Phi$. With $v \in W \setminus R_0$, the vertex-set of the initial k-face F_k is $V_k := v\mathbf{G}_k$; a general k-face has vertex-set $V_k\Theta$ for some $\Theta \in \mathbf{G}$.

We think of the partially ordered family P of these vertex-sets as the realization of the abstract regular m-polytope whose group is $\boldsymbol{G}$. Moreover, we now denote by $\mathcal{P}$ the space of congruence classes of these realizations, as well as the abstract regular polytope with group $\boldsymbol{G}$. In this notation, we therefore write $P \in \mathcal{P}$.

We now call the realization faithful if P, as a partially ordered set, is isomorphic to $\mathcal{P}$. The earlier concept is renamed vertex-faithful.

Faces and cofaces

It is obvious that a realization of a regular polytope $\mathcal{P}$ contains a realization of each of its faces, with an induced subgroup of the whole symmetry group $\mathbf{G}$ acting on it.

In fact, the same is true of the vertex-figure, and more generally of cofaces and sections. For the vertex-figure, if v is the initial vertex of the realization P and $w := vR_0$, then we have two choices. The narrow vertex-figure has initial vertex $\frac{1}{2}(v + w)$, while the initial vertex of the broad vertex-figure is w itself; in both cases, $\mathbf{G}_0 = \langle R_1, \ldots, R_{m-1} \rangle$ acts in the natural way.

The convenience of the 'broad' definition is that the vertices of the vertex-figure form a subset of the original vertex-set.

However, for more general cofaces (and sections), the correct approach is to take the 'narrow' path. For the k-coface, we apply Wythoff's construction with group $\langle R_{m-k}, \ldots, R_{m-1} \rangle$ and initial vertex some

$$w_{m-k} \in R_0 \cap \cdots \cap R_{m-k-1} \cap R_{m-k+1} \cap \cdots \cap R_{m-1}.$$

The case $k = m - 1$ is, of course, just that of the narrow vertex-figure, as previously defined.

Now, the induced realization of a section is just that of the appropriate face of a coface.

Rank and dimension

For the next part, it is convenient to take the ambient space $\mathbb{A}$ to be a spherical space $\mathbb{S}$ or euclidean space $\mathbb{E}$, with its instrinsic dimension. In the spherical case, we ultimately add 1 to each dimension.

There are two main results, both proved by induction using vertex-figures.

Theorem

If P is a faithful realization of $\mathcal{P}$, then $\dim P \geqslant \operatorname{rank} \mathcal{P} - 1$.

The mirror vector of the realization is

$$\dim(R_0, \ldots, R_{m-1}) := (\dim R_0, \ldots, \dim R_{m-1}).$$

Theorem

For a faithful realization of $\mathcal{P}$,

$$\dim(R_0, R_1, \ldots, R_{m-2}, R_{m-1}) \geqq (0, 1, \ldots, m-2, m-2).$$

Full rank

Say that the faithful realization P of $\mathcal{P}$ (with ambient space $\mathbb{A}$ as before) is of full rank if $\dim P = \operatorname{rank} \mathcal{P} - 1$.

Theorem

The mirror vector $\dim(R_0, \ldots, R_{m-1})$ *of a faithful realization of full rank satisfies*

$$\dim R_j = \begin{cases} j \text{ or } m-2, & \text{if } j = 0, \ldots, m-3, \\ m-2, & \text{if } j = m-2 \text{ or } m-1. \end{cases}$$

The key here is that the vertex-figure Q must also be of full rank, lying in a sphere $\mathbb{S}$ in $\mathbb{A}$ centred at the initial vertex v of P. For $j \geqslant 1$, R_j is spanned by v and the corresponding mirror S_{j-1} of Q, while the edge-figure also being of full rank forces R_0 to be a point or hyperplane. (If $x \in R_0$ is not the mid-point of the initial edge, then $x\langle R_2, \ldots, R_{d-1}\rangle$ spans a hyperplane.)

Finite polytopes

Until further notice, all regular polytopes $\mathcal{P}$ will be finite; we shall say a little about apeirotopes at the end of this part.

Remark

Actually, apeirotopes with finitely many vertices will also fit in here.

We denote by n the number of vertices and r the number of non-trivial diagonal classes of $\mathcal{P}$. For each pure realization Ψ, let $d = d(\Psi)$ be the corresponding dimension, $W = W(\Psi)$ the Wythoff space and $w = w(\Psi) := \dim W$ its dimension.

In fact, what is more important is the essential Wythoff space W^*, and its dimension $w^* := \dim W^*$. This factors out by non-scalar centralizers of $\mathbf{G}$ in the orthogonal group $\mathbf{O}$, and divides w by the character norm – which is $1, 2$ or 4 – to yield w^*. However, we shall not encounter examples of this phenomenon here, though we note it in what follows.

Fundamental relations

We have the following basic relations for pure realizations. We write $\overline{W}$ for the Wythoff space of the simplex realization E with vertex-set the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{E}^n$, and $\overline{w} := \dim \overline{W}$.

Theorem

$$\begin{aligned}
\textstyle\sum_{\Psi} \; w^*(\Psi)d(\Psi) &= n, \\
\textstyle\sum_{\Psi} \; \tfrac{1}{2}w^*(\Psi)(w^*(\Psi)+1) &= r+1, \\
\textstyle\sum_{\Psi} \; w^*(\Psi)w(\Psi) &= \overline{w},
\end{aligned}$$

where the sums are over the irreducible representations Ψ of $\boldsymbol{G}$.

Remark

Bear in mind the trivial realization $\{1\}$, for which $d = w = w^* = 1$.

The core point is that each W^* contributes w^* pure components to E (accounting for n and $\overline{w}$), but $\frac{1}{2}w^*(w^*+1)$ to the dimension of the realization cone (diagonal vectors are quadratic functions of linear combinations of initial vertices in W^*).

Cosine vectors

Recall that the diagonal classes of $\mathcal{P}$ split its vertex-set into layers from the initial vertex e and that, if there are ℓ_s vertices in layer s, then $\Lambda := (\ell_0, \ldots, \ell_r)$ is the layer vector of $\mathcal{P}$. As before, the entry $\ell_0 = 1$ always corresponds to the trivial diagonal; usually, ℓ_1 corresponds to the initial edge.

We now normalize, so that the vertices of all realizations P lie on some unit sphere. The normalized realizations of $\mathcal{P}$ form the realization domain $\mathcal{N} = \mathcal{N}(\mathcal{P})$ of $\mathcal{P}$.

Also recall that the cosine vector $\Gamma = \Gamma(P) = (\gamma_0, \gamma_1, \ldots, \gamma_r)$ is given by $\gamma_k := \langle v, x \rangle$, where $\{v, x\}$ represents the kth diagonal class from the inital vertex v. The initial entry is $\gamma_0 := \langle v, v \rangle = 1$. Thus cosine vectors Γ are identified with realizations in $\mathcal{N}$.

Blends

First observe that, in the normalized case, diagonal and cosine vectors are related by

$$\delta_k = 2(1 - \gamma_k).$$

A blend $\mu P \# \nu Q$ of two normalized realizations P, Q of $\mathcal{P}$ will also be normalized just when $\mu^2 + \nu^2 = 1$. In terms of cosine vectors (and diagonal vectors as well), this corresponds to taking convex combinations. For the realization domain $\mathcal{N}$, this says

Theorem

The realization domain $\mathcal{N}$ of a regular polytope $\mathcal{P}$ is a compact convex set of dimension r, the number of non-trivial diagonal classes of $\mathcal{P}$.

We call a realization $P \in \mathcal{P}$ centred if the origin o is the centroid of its vertices.

Theorem

If the centred realization $P \in \mathcal{P}$ has cosine vector $\Gamma = (\gamma_0, \ldots, \gamma_r)$, then

$$\langle \Lambda, \Gamma \rangle = \sum_{s=0}^{r} \ell_s \gamma_s = 0.$$

The trivial realization $P_0 = \{1\}$ has cosine vector $\Gamma_0 = (1^{r+1})$, so that $\langle \Lambda, \Gamma_0 \rangle = |\Lambda| = n$, the number of vertices of $\mathcal{P}$. More generally, $\langle \Gamma, \Lambda \rangle / |\Lambda|$ is the coefficient of Γ_0 in the expression of Γ as a convex combination of pure cosine vectors.

Dimension equation

Theorem

If the simplex realization $E = T^{n-1}$ of $\mathcal{P}$ is decomposed into pure components $P_0, \ldots, P_k$, where P_j has dimension d_j and cosine vector Γ_j for $j = 0, \ldots, k$, then

$$\sum_j d_j \Gamma_j = n(1, 0^r).$$

This follows from the fact that the radius ρ_j of P_j satisfies $\rho_j^2 = d_j/n$. Of course, $(1, 0^r)$ is the cosine vector of E.

Remark

If $\dim P = d$, then $P \otimes P$ has a non-trivial component of dimension at most

$$\tfrac{1}{2}d(d+1) - 1 = \tfrac{1}{2}(d-1)(d+2).$$

Central symmetry

The analogous expresssion for centrally symmetric polytopes is

Theorem

If the cross-polytope realization X^n of a centrally symmetric polytope $\mathcal{P}$ with $2n$ vertices is decomposed into pure components $P_1, \ldots, P_k$, where P_j has dimension d_j and cosine vector Γ_j for $j = 1, \ldots, k$, then

$$\sum_j d_j \Gamma_j = n(1, 0^{r-1}, -1).$$

The convention for central symmetry is that the last diagonal class is that to the antipodal vertex; occasionally we go against this. The sum for the realizations of $\mathcal{P}/2$ will now be $n(1, 0^{r-1}, 1)$, corresponding to the small simplex realization.

Simplex

We first give some basic examples. The d-simplex $\mathcal{T}^d := \{3^{d-1}\}$ has only one non-trivial diagonal class, and so its layer vector is $\Lambda = (1, d)$. Thus the centred simplex $P_1 = T^d$ itself must have cosine vector

$$\Gamma_1 = (1, -\tfrac{1}{d}),$$

as may be seen directly from the geometry.

There is only one non-trivial product, namely, $T^d \otimes T^d$, with cosine vector

$$\Gamma_1^2 = (1, \tfrac{1}{d^2}) = \tfrac{1}{d}\Gamma_0 + \tfrac{d-1}{d}\Gamma_1.$$

The invariable convention is that Γ_0 is the cosine vector of the henagon $\{1\}$ as the trivial realization; here, $\Gamma_0 = (1, 1)$. Note that the coefficient of Γ_0 is indeed given by

$$\langle \Lambda, \Gamma_1^2 \rangle / |\Lambda| = \big(1 + d \cdot \tfrac{1}{d^2}\big)/(d+1) = \tfrac{1}{d}.$$

As a non-regular example, consider the cartesian product $P_1 := T^d \times T^d$ of centred congruent d-simplices, with symmetry group $(\mathbf{S}_{d+1} \times \mathbf{S}_{d+1}) \rtimes \boldsymbol{C}_2$. Then P_1 itself (suitably scaled) has cosine vector

$$\Gamma_1 = (1, \tfrac{d-1}{2d}, -\tfrac{1}{d});$$

there are $2d$ vertices adjacent to the initial one. (Why can this be thought of as $\Gamma_1 = \frac{1}{2}\big((1, -\frac{1}{d}, -\frac{1}{d}) + (1, 1, -\frac{1}{d})\big)$?).

If we take the vertices of the two copies to be the unit vectors $x_0, \ldots, x_d$ and $y_0, \ldots, y_d$, then the $x_j \otimes y_k$ clearly give the vertices of another realization P_2 in $\mathbb{E}^{d^2}$. Its cosine vector will be

$$\Gamma_2 = (1, -\tfrac{1}{d}, \tfrac{1}{d^2}).$$

(This is $(1, -\frac{1}{d}, -\frac{1}{d})(1, 1, -\frac{1}{d})$ – again, can you see why?)

Cross-polytope

The centrally symmetric d-cross-polytope $\mathcal{X}^d := \{3^{d-2}, 4\}$ has layer vector $\Lambda = (1, 2(d-1), 1)$. Its standard centrally symmetric realization $P_2 = X^d$ has cosine vector $\Gamma_2 = (1, 0, -1)$.

Identifying opposite vertices to give $\mathcal{X}^d/2$ in effect collapses $\mathcal{X}^d$ onto its facet $\mathcal{T}^{d-1}$. We thus have the small simplex realization $P_1 = T^{d-1}$, with cosine vector $\Gamma_1 = (1, -\frac{1}{d-1}, 1)$.

For the products $P_1 \otimes P_2$ and $P_2 \otimes P_2$, we have

$$\begin{aligned} \Gamma_1 \Gamma_2 &= \Gamma_2 \quad (\implies T^{d-1} \otimes X^d = X^d), \\ \Gamma_2^2 &= \tfrac{1}{d}\Gamma_0 + \tfrac{d-1}{d}\Gamma_1. \end{aligned}$$

Polygons

For $p \geqslant 3$, the p-gon $\{p\}$ has $\lfloor \frac{1}{2}p \rfloor$ non-trivial pure realizations $\{\frac{p}{s_j}\}$, with $s_j = 1, \ldots, \lfloor \frac{1}{2}p \rfloor$. If p is even, then the last of these is the digon $\{2\}$.

In taking the product of two polygons, we can regard them as covered by a common one. We then have (suppressing the scaling constants $\frac{1}{\sqrt{2}}$)

Theorem

For $p \geqslant q \geqslant 2$ rational,

$$\{p\} \otimes \{q\} = \begin{cases} \left\{\dfrac{pq}{p-q}\right\} \# \left\{\dfrac{pq}{p+q}\right\}, & \text{if } p > q, \\ \left\{\dfrac{p}{2}\right\} \# \{1\}, & \text{if } p = q. \end{cases}$$

Icosahedron

We give two further examples. The convex regular icosahedron $P_2 := \{3, 5\} \in \boldsymbol{\{3, 5\}}$, with layer vector $\Lambda = (1, 5, 5, 1)$, has vertices all even permutations of $(\pm\tau, \pm 1, 0)$. Its cosine vector is thus

$$\Gamma_2 = (1, \tfrac{1}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}, -1).$$

The great icosahedron $P_3 := \{3, \frac{5}{2}\}$ (with the same vertices) similarly has cosine vector

$$\Gamma_3 = (1, -\tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, -1).$$

There are several ways of finding the last cosine vector.

- Since the realization can only be of $\{3,5\}/2 = \{3,5:5\}$, whose vertices are those of the 5-simplex, it must be

$$\Gamma_1 = (1, -\tfrac{1}{5}, -\tfrac{1}{5}, 1).$$

- Applying the dimension equation, with $d_1 = 12 - 1 - 3 - 3 = 5$ gives

$$\Gamma_1 = \tfrac{1}{5}\bigl(12(1,0,0,0) - \Gamma_0 - 3\Gamma_2 - 3\Gamma_3\bigr) = (1, -\tfrac{1}{5}, -\tfrac{1}{5}, 1).$$

- Since the simple group $\boldsymbol{A}_5$ has no non-trivial representations in $\mathbb{R}$ or $\mathbb{E}^2$, the non-trivial component of $P_2 \otimes P_2$ must be pure. Using the layer equation, this gives

$$\Gamma_2^2 = (1, \tfrac{1}{5}, \tfrac{1}{5}, 1) = \tfrac{1}{3}\Gamma_0 + \tfrac{2}{3}\Gamma_1.$$

Dodecahedron

The regular dodecahedron $\{5, 3\}$ with 20 vertices has layer vector $\Lambda = (1, 3, 6, 6, 3, 1)$. The hemi-dodecahedron $\{5, 3\}/2 = \{5, 3 : 5\}$ has 2 non-trivial diagonal classes, leaving 3 corresponding to faithful realizations. Two of these are the convex dodecahedron $P_3 = \{5, 3\}$ and great stellated dodecahedron $P_4 = \{\frac{5}{2}, 3\}$, with cosine vectors

$$\Gamma_3 = (1, \tfrac{\sqrt{5}}{3}, \tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{\sqrt{5}}{3}, -1),$$
$$\Gamma_4 = (1, -\tfrac{\sqrt{5}}{3}, \tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{\sqrt{5}}{3}, -1).$$

The final pure faithful realization P_5 must thus have dimension $d_5 = 10 - 3 - 3 = 4$, and – from the dimension equation – cosine vector

$$\Gamma_5 = \tfrac{1}{4}\big(10(1, 0, 0, 0, -1) - 3\Gamma_3 - 3\Gamma_4\big) = (1, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, -1).$$

Just as for the icosahedron, one pure realization P_2 of $\{5, 3 : 5\}$ will be the non-trivial component of $P_3 \otimes P_3$ of dimension $d_2 = 5$, giving $\Gamma_3^2 = \frac{1}{3}\Gamma_0 + \frac{2}{3}\Gamma_2$ with

$$\Gamma_2 = (1, \tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, 1).$$

Thus the final pure realization P_1 has dimension $d_1 = 10 - 1 - 5 = 4$, and cosine vector

$$\Gamma_1 = \tfrac{1}{4}\big(10(1, 0, 0, 0, 0, 1) - \Gamma_0 - 5\Gamma_2\big) = (1, -\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}, -\tfrac{2}{3}, 1).$$

Among various product relations is

$$\Gamma_3\Gamma_4 = \tfrac{8}{9}\Gamma_1 + \tfrac{1}{9}\Gamma_2 \implies P_3 \otimes P_4 = \tfrac{2\sqrt{2}}{3}P_1 \# \tfrac{1}{3}P_2;$$

observe that $P_3 \otimes P_4$ is centred.

In summary, then, the cosine matrix of $\{5, 3\}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & 1 \\ 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \\ 1 & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{5}}{3} & -1 \\ 1 & -\frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{\sqrt{5}}{3} & -1 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \end{bmatrix}$$

Apeirotopes

The realization theory of regular apeirotopes is more complicated, and so we shall say very little about it (the finite theory applied to vertex-figures will suffice). However, as an illustration, we give a simple example.

Let

$$0 < \vartheta_1 < \cdots < \vartheta_k \leqslant \pi,$$

and define

$$x_n = \big(n, \cos(n\vartheta_1), \sin(n\vartheta_1), \ldots, \cos(n\vartheta_k), \sin(n\vartheta_k)\big) \in \mathbb{E}^{2k+1}$$

for $n \in \mathbb{Z}$. For each k and choice of ϑ_j, this gives a (discrete) regular apeirogon $\{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$; thus the realization space of the (abstract) apeirogon $\{\infty\}$ has uncountably infinite (algebraic) dimension.

Another curiosity is the following.

Theorem

If an abstract regular apeirotope $\mathcal{P}$ has a discrete faithful realization P whose translation group is full-dimensional (in its ambient space), then $\mathcal{P}$ has a faithful realization with no translations in its symmetry group.

What happens here is that the translational symmetries of P are destroyed by turning them into irrational apeirogonal symmetries, as in the previous example.

Remark

If we drop discreteness, then we can find faithful realizations of hyperbolic honeycombs in euclidean spaces, or even on spheres.