4-dimensional polyhedra

Blended polyhedra

The first step is to note the blended polyhedra $P \# \{2\}$, where P is any one of the nine classical polyhedra or their Petrials. These are all properly polyhedral.

Remark

These blended polyhedra fall into isomorphic pairs, since

$$P \# \{2\} \cong P \diamond \{2\} := P^{\zeta} \# \{2\}.$$

Each vertex of $P \# \{2\}$ is of the form (v, ε), with $v \in \operatorname{vert} P$ and $\varepsilon = \pm 1$. The isomorphism is then given by

$$(v, \varepsilon) \longleftrightarrow (\varepsilon v, \varepsilon).$$

Mirror vectors

As with the 3-dimensional apeirohedra, mirror vectors play a crucial rôle in the classification of the pure 4-dimensional regular polyhedra. The possible vectors are

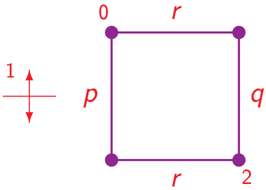
- $(3, 2, 3)$, $(2, 2, 3)$, $(1, 2, 3)$;
- $(2, 3, 2)$;
- $(2, 2, 2)$.

The reason for the grouping is as follows. In the first class, the basic mirror vector is $(3, 2, 3)$; polyhedra with the other mirror vectors are derived from basic ones by Petriality π or ζ, or both. The other two classes are preserved by π and ζ.

The third class is of additional interest, in that its members are handed, meaning that their symmetry groups consist of rotations alone.

Mirror vector $(3, 2, 3)$

These polyhedra are derived by twisting a diagram $\mathcal{D}_1(p, q; r)$; they have planar faces and holes, and skew vertex-figures. A label j indicates the generator R_j of the symmetry group; thus R_1 is a twist.



The diagram $\mathcal{D}_1(p, q; r)$

Here, p cannot be a fraction with an even denominator, but q can be. In the latter case, the derived polyhedron is of the form $P = Q^{\varpi}$ for some classical regular 4-polytope Q.

Examples

From $\mathcal{D}_1(2,3;3)$ (the diagram of the 4-simplex) are derived the dual pair $\{4, \frac{6}{1,3} \mid 3\} \cong \boldsymbol{\{4,6 \mid 3\}}$ and $\{6, \frac{4}{1,2} \mid 3\} \cong \boldsymbol{\{6,4 \mid 3\}}$. The former doubly-covers $\{4, \frac{6}{2,3} \mid 3\} = \{3,3,3\}^{\varpi} \cong \boldsymbol{\{4,6 : 5 \mid 3\}}$, derived from $\mathcal{D}_1(2,3;\frac{3}{2})$; this has no geometric dual (in $\mathbb{E}^4$).

In most cases, applications of π or ζ are not of much interest. However, here we have

$$\{4, \tfrac{6}{1,3} \mid 3\}^{\zeta} = \{\tfrac{4}{1,2}, \tfrac{6}{1,3} : \tfrac{5}{1,2}\} \cong \boldsymbol{\{4,6 : 5\}},$$
$$\{6, \tfrac{4}{1,2} \mid 3\}^{\zeta} = \{\tfrac{6}{1,3}, \tfrac{4}{1,2} : \tfrac{5}{1,2}\} \cong \boldsymbol{\{6,4 : 5\}}.$$

These are again duals, but not geometrically (their mirror vectors are $(1,2,3)$). The former also doubly-covers $\{4, \frac{6}{2,3} \mid 3\}$.

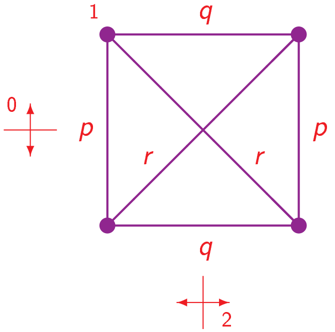
From $\mathcal{D}_1(2,4;3)$ (the diagram of the 24-cell) is derived a pretty family:

$$
\begin{tikzcd}
\{8, \frac{4}{1,2} \mid 3\} \arrow[r, leftrightarrow, "\delta"] & \{4, \frac{8}{1,4} \mid 3\} \arrow[d, leftrightarrow, "\varphi_3"'] & & \\
& \{8, \frac{8}{3,4} \mid 3\} \arrow[r, leftrightarrow, "\delta"] & \{\frac{8}{3}, \frac{8}{1,4} \mid 3\} \arrow[d, leftrightarrow, "\varphi_3"'] & \\
& & \{4, \frac{8}{3,4} \mid 3\} \arrow[r, leftrightarrow, "\delta"] & \{\frac{8}{3}, 4 \mid 3\}
\end{tikzcd}
$$

These also employ the diagrams $\mathcal{D}_1(4, \frac{4}{3}; 3)$ and $\mathcal{D}_1(2, \frac{4}{3}; 3)$; note the use of the facetting operator φ_3.

Mirror vector $(2,3,2)$

These polyhedra are derived by twisting a diagram $\mathcal{D}_2(p,q,r)$; they have skew faces and Petrie polygons (Petriality π interchanges p and r), and planar vertex-figures. Now q cannot be a fraction with an even denominator.



The diagram $\mathcal{D}_2(p,q,r)$

Examples

Potentially, a diagram $\mathcal{D}_2(p, q, r)$ can gives rise to six polyhedra, related by duality δ and Petriality π. In practice, the family is usually smaller, because – to obtain all six – none of p, q, r can be a fraction with an even denominator. However, we obtain all six possibilities from $\mathcal{D}_2(3, 4, \frac{4}{3})$:

$$\begin{array}{ccccc}
\{\frac{6}{1,3}, 8 : \frac{8}{3,4}\} & \overset{\delta}{\longleftrightarrow} & \{\frac{8}{1,4}, 6 : \frac{8}{3,4}\} & \overset{\pi}{\longleftrightarrow} & \{\frac{8}{3,4}, 6 : \frac{8}{1,4}\} \\
\pi \updownarrow & & & & \delta \updownarrow \\
\{\frac{8}{3,4}, 8 : \frac{6}{1,3}\} & \overset{\delta}{\longleftrightarrow} & \{\frac{8}{1,4}, \frac{8}{3} : \frac{6}{1,3}\} & \overset{\pi}{\longleftrightarrow} & \{\frac{6}{1,3}, \frac{8}{3} : \frac{8}{1,4}\}
\end{array}$$

Note the occurrence of $(\pi\delta)^3 = \varepsilon$, the identity. The symbols here are abbreviated, and do not indicate the combinatorial types.

If we apply η to a polyhedron of type $\{4, q\}$ with mirror vector $(3, 2, 3)$, then the result has mirror vector $(2, 3, 2)$. So, for example,

$$\begin{aligned}
\{4, \tfrac{6}{1,3} \mid 3\}^{\eta} &= \{\tfrac{6}{1,3}, 6 : \tfrac{6}{2,3}\}, \\
\{4, \tfrac{8}{1,4} \mid 3\}^{\eta} &= \{\tfrac{8}{1,4}, 8 : \tfrac{6}{2,3}\}, \\
\{4, \tfrac{8}{3,4} \mid 3\}^{\eta} &= \{\tfrac{8}{3,4}, \tfrac{8}{3} : \tfrac{6}{2,3}\}.
\end{aligned}$$

The first of these is derived from the diagram $\mathcal{D}_2(3, 3, \frac{3}{2})$. More generally, under η we have

$$\mathcal{D}_1(2, q; r) \overset{\eta}{\longmapsto} \mathcal{D}_2(q, q, \tfrac{r}{2}).$$

Observe that the latter two polyhedra above therefore belong to a different diagram from the previous family, namely, $\mathcal{D}_2(4, 4, \frac{3}{2})$.

In the same way,

$$\mathcal{D}_2(2, q, r) \xmapsto{\eta} \mathcal{D}_1(q, q; r);$$

however, we cannot have $r = 2$ here, since any resulting polyhedron would have digonal holes, and so would degenerate.

Remark

It is worth commenting on the tori of type $\{4, 4\}$, which are associated with diagrams $\mathcal{D}_1(2, 2; r)$ and $\mathcal{D}_2(2, 2, r)$. If $r = 2s$ is an even integer, then

$$\{4, \tfrac{4}{1,2} \mid 2s\}^\eta = \{\tfrac{4}{1,2}, 4 : 2s\}, \quad \{\tfrac{4}{1,2}, 4 : 2s\}^\eta = \{4, \tfrac{4}{1,2} \mid s\}.$$

On the other hand, when r is odd,

$$\{4, \tfrac{4}{1,2} \mid r\} \xleftrightarrow{\eta} \{\tfrac{4}{1,2}, 4 \mid r\}.$$

Mirror vector $(2, 2, 2)$

We tackle this class through quaternions. First note that $R\colon \mathbf{x} \mapsto \bar{\mathbf{a}}\mathbf{x}\mathbf{b}$ is an involution just when $\mathbf{a}, \mathbf{b}$ are pure imaginary (ignoring the trivial case $\{\mathbf{a}, \mathbf{b}\} = \{\pm 1\}$). So, for the group of a regular polyhedron P, we look for generators R_j of this form, each corresponding to a pair $(\mathbf{a}_j, \mathbf{b}_j)$ for $j = 0, 1, 2$.

The group $\langle R_1, R_2 \rangle$ of the vertex-figure cannot contain a double rotation, and thus we may assume that

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \langle \mathbf{b}_1, \mathbf{b}_2 \rangle = -\cos\left(\frac{\pi}{q}\right)$$

for some q. On the other hand, $R_0 R_1$ must be a double rotation, otherwise P is classical. Thus we have $r_1 \neq r_2$ such that

$$\langle \mathbf{a}_0, \mathbf{a}_1 \rangle = -\cos\left(\frac{\pi}{r_1}\right), \quad \langle \mathbf{b}_0, \mathbf{b}_1 \rangle = -\cos\left(\frac{\pi}{r_2}\right).$$

Bearing in mind the connexion between quaternions and rotations in $\mathbb{E}^3$, this associates P with a pair of spherical triangles $(r_j, 2, q)$; the rotations will be about axes orthogonal to the planes of these triangles. Taking $r_1 < r_2$ for the moment, the face of P will be a polygon $\{\frac{p}{s,t}\}$, with

$$\frac{s}{p} = \frac{1}{2}\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big), \quad \frac{t}{p} = \frac{1}{2}\Big(\frac{1}{r_1} + \frac{1}{r_2}\Big).$$

Now the Petrial Q of P will be a polyhedron of the same kind, with r_j replaced by h_j, given by

$$\cos^2\left(\frac{\pi}{r_j}\right) + \cos^2\left(\frac{\pi}{q}\right) + \cos^2\left(\frac{\pi}{h_j}\right) = 1.$$

An easy way to see this last is to choose coordinates (as we may) so that

$$\begin{aligned}
\mathbf{a}_0 &= \mathbf{i},\\
\mathbf{a}_1 &= \cos\left(\frac{\pi}{r_1}\right)\mathbf{i} + \cos\left(\frac{\pi}{q}\right)\mathbf{j} + \cos\left(\frac{\pi}{h_1}\right)\mathbf{k},\\
\mathbf{a}_2 &= \mathbf{j}.
\end{aligned}$$

We denote the resulting regular polyhedron (if it exists) by

$$P = \{r_1, q : h_1\} \bowtie \{r_2, q : h_2\}.$$

Not all choices of compatible spherical triangles do yield polyhedra, as we shall see.

A better way to calculate such things as edge-lengths is to make a different choice:

$$\begin{aligned}
\mathbf{a}_0 &= \alpha_1 \mathbf{i} + \alpha_2 \mathbf{j}, \qquad \mathbf{b}_0 = \beta_1 \mathbf{i} + \beta_2 \mathbf{j}, \\
\mathbf{a}_1 &= \mathbf{b}_1 = \sin(\pi/q)\mathbf{j} - \cos(\pi/q)\mathbf{k}, \\
\mathbf{a}_2 &= \mathbf{b}_2 = \mathbf{k}.
\end{aligned}$$

We now have

$$\alpha_1 = \frac{\cos(\pi/h_1)}{\sin(\pi/q)}, \quad \alpha_2 = -\frac{\cos(\pi/r_1)}{\sin(\pi/q)},$$

and similarly for β_1, β_2. (We do not insist that $r_j > 1$, so that $\cos(\pi/r_j) < 0$ is permitted, and so on.)

Different liftings of the rotation groups into $\mathbf{Q}$ change the signs of the $\mathbf{a}_j, \mathbf{b}_j$. We may impose these sign changes on the $\mathbf{a}_j$. Changing that of $\mathbf{a}_0$ is ζ, and replaces (r_1, h_1) by (r_1', h_1'), with

$$\frac{1}{r_1} + \frac{1}{r_1'} = 1, \quad \frac{1}{h_1} + \frac{1}{h_1'} = 1.$$

Further, $\mathbf{a}_1$ and $\mathbf{a}_2$ must change signs together (to preserve $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$); doing that replaces r_1 by r_1', and hence changing all signs replaces h_1 by h_1'.

A striking feature is that the left and right groups of quaternions can be quite different. Indeed, it is only when they are the same that we can run into problems.

A bad example

For a putative polyhedron $P = \{3, 5 : \frac{5}{3}\} \bowtie \{\frac{5}{2}, 5 : 3\}$, we can take

$$\begin{aligned}
\mathbf{a}_0 &= \mathbf{k}, \quad \mathbf{b}_0 = \mathbf{j},\\
\mathbf{a}_1 &= \mathbf{b}_1 = -\tfrac{1}{2}(\tau\mathbf{i} + \tau^{-1}\mathbf{j} + \mathbf{k}),\\
\mathbf{a}_2 &= \mathbf{b}_2 = \mathbf{i}.
\end{aligned}$$

Since (up to scaling) 1 is the initial vertex, it is not hard to see that we obtain the icosians $\widehat{\mathbf{G}}$ as the vertex-set. However, since the group here is just the rotation group $[3, 3, 5]^+$ of the 600-cell, we should obtain $7200/10 = 720$ vertices; in other words, the vertices coincide in sixes, and so P degenerates.

Remark

In fact, most non-self-Petrie polyhedra of this kind are degenerate.

It turns out that the self-Petrie polyhedra are precisely the ones which can be obtained by a rather strange mixing operation σ^* from those of type $\{4, q\}$ with mirror vector $(3, 2, 3)$. This operation is related to halving η by

$$\sigma^* := \pi^* \eta \pi^* \colon (s_0, s_1, s_2) \mapsto (s_0 s_1 s_0, s_2, (s_0 s_1)^2) =: (r_0, r_1, r_2).$$

Here, $\pi^* := \delta\pi\delta = \pi\delta\pi$ is the dual operation to π, so that

$$\pi^* \colon (s_0, s_1, s_2) \mapsto (s_0, s_1, s_0 s_2) =: (r_0, r_1, r_2).$$

Petriality π replaces r_0 by $r_0 r_2 = s_1$; this is also got on conjugating the generators by s_0, an observation to be borne in mind.

Enumeration

We will not actually enumerate the polyhedra in this class; instead, we point the way towards the enumeration. In the earlier discussion of quaternions, we mentioned that the groups of interest were

$$\widehat{\mathbf{D}}_n, \quad \widehat{\mathbf{C}}, \quad \widehat{\mathbf{G}}.$$

These correspond to the triples $\{r, q, h\} = \{2, q, q''\}$, $\{3, 3, 4\}$ and $\{3, 5, \frac{5}{2}\}$, with $q > 2$ (since $\{q\}$ is a vertex-figure) and q'' defined as usual by

$$\frac{1}{q} + \frac{1}{q''} = \frac{1}{2};$$

moreover, n is the common numerator of q and q'' (and so even).

The group $\widehat{\mathbf{A}}$ does not occur here, since the corresponding rotation group is not generated by involutions.

The cases involving the icosians are of most interest, because as well as $\widehat{\mathbf{G}}$ we have the isomorphic, but not congugate, group $\widehat{\mathbf{G}}^\dagger$, obtained by changing the sign of $\sqrt{5}$. The subgroup consisting of the mappings $\mathbf{x} \mapsto \overline{\mathbf{b}}^\dagger \mathbf{x}\mathbf{b}$, with $\mathbf{b} \in \widehat{\mathbf{G}}$, has order 60; it is just the alternating group $\boldsymbol{A}_5$.

Allowing the various sign changes then gives a family of four polyhedra, all familiar except possibly the last (the edge-graph of this polyhedron is the generalized Petersen graph $G(10,3)$, just as $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$ corresponds to $G(5,3)$):

$$\{5,3:\tfrac{5}{3}\} \bowtie \{\tfrac{5}{3},3:5\} = \{\tfrac{5}{1,2},3:\tfrac{5}{1,2}\},$$
$$\{5,3:\tfrac{5}{2}\} \bowtie \{\tfrac{5}{3},3:5\} = \{\tfrac{5}{1,2},3:\tfrac{10}{1,3}\},$$
$$\{5,3:\tfrac{5}{3}\} \bowtie \{\tfrac{5}{2},3:5\} = \{\tfrac{10}{1,3},3:\tfrac{5}{1,2}\},$$
$$\{5,3:\tfrac{5}{2}\} \bowtie \{\tfrac{5}{2},3:5\} = \{\tfrac{10}{1,3},3:\tfrac{10}{1,3}\}.$$

We shall say little about the other polyhedra in these families. For example, if the left group is $\widehat{\mathbf{C}}$ and the right group is $\widehat{\mathbf{G}}$, then we obtain (in all) 16 polyhedra of type $\{40, 3 : 40\}$ (obviously far from universal), with 480 vertices and symmetry group of order 2880.

There are infinitely many polyhedra

$$\{2, q : q''\} \bowtie \{q'', q : 2\} = \{\tfrac{2s}{t,s-t}, q : \tfrac{2s}{t,s-t}\},$$

where $q = s/t$ in lowest terms.

Apart from these, since we can then only have $q = 3, 4, 5, \frac{5}{2}$, the other families are all finite.