Euclidean spaces and isometries

Euclidean spaces

We take for granted familiarity with euclidean spaces and their basic properties. The inner (or scalar) product of $x, y \in \mathbb{E}$ is denoted $\langle x, y\rangle = \langle y, x\rangle$, and the corresponding norm of x is $\|x\| = \sqrt{\langle x, x\rangle}$. The euclidean distance between x and y is $\|x - y\|$.

Remark

The linear hull of $S \subseteq \mathbb{E}$ – that is, the set of all linear combinations of points of S – is denoted $\operatorname{lin} S$.

Every k-dimensional linear subspace $\mathbb{L} \leqslant \mathbb{E}$ has an orthonormal basis $\{u_1, \ldots, u_k\}$, so that

$$\langle u_i, u_j\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Affine structure

An affine combination of points of $\mathbb{E}$ is one of the form

$$\lambda_0 a_0 + \cdots + \lambda_k a_k, \quad \text{such that } \lambda_0 + \cdots + \lambda_k = 1.$$

The affine hull of $S \subseteq \mathbb{E}$ is the set of affine combinations of points of S, and is denoted $\operatorname{aff} S$; we also say that S spans $\operatorname{aff} S$ affinely. A set $\mathbb{A}$ is an affine subspace or flat if $\operatorname{aff} \mathbb{A} = \mathbb{A}$. For example, a line is of the form

$$ab := \{(1-\lambda)a + \lambda b \mid \lambda \in \mathbb{R}\}.$$

Remark

A non-empty affine subspace $\mathbb{A}$ is a translate $\mathbb{A} = \mathbb{L} + a$ of a unique linear subspace $\mathbb{L}$; here, $a \in \mathbb{A}$ is any point. Its dimension is $\dim \mathbb{A} := \dim \mathbb{L}$; thus $\dim \emptyset = -1$ is a natural convention.

A set $S \subseteq \mathbb{E}$ is affinely dependent if $x \in \operatorname{aff}(S \setminus \{x\})$ for some $x \in S$. The condition for this is that there are $a_0, \ldots, a_k \in S$ and $\lambda_0, \ldots, \lambda_k \in \mathbb{R}$, not all 0, such that

$$\lambda_0 a_0 + \cdots + \lambda_k a_k = o, \quad \lambda_0 + \cdots + \lambda_k = 0.$$

Obviously, a set which is not affinely dependent is affinely independent. An affinely independent subset B of an affine subspace $\mathbb{A}$ such that $\mathbb{A} = \operatorname{aff} B$ is an affine basis of $\mathbb{A}$. Each affine basis of $\mathbb{A}$ contains $\dim \mathbb{A} + 1$ points; indeed, $\{a_0, \ldots, a_k\}$ is an affine basis of $\mathbb{A}$ if and only if $\{a_1 - a_0, \ldots, a_k - a_0\}$ is a (linear) basis of the linear subspace $\mathbb{L} = \mathbb{A} - a_0$.

Convexity

A positive combination of points of $\mathbb{E}$ is one of the form

$$\lambda_1 a_1 + \cdots + \lambda_k a_k, \quad \text{such that } \lambda_1, \ldots, \lambda_k \geqslant 0.$$

The positive hull of $S \subseteq \mathbb{E}$ is the set of positive combinations of points of S, and is denoted $\operatorname{pos} S$. A set C is a (convex) cone if $\operatorname{pos} C = C$.

A convex combination is one which is both affine and positive. The convex hull of $S \subseteq \mathbb{E}$ is the set of convex combinations of points of S, and is denoted $\operatorname{conv} S$. A set C is convex if $\operatorname{conv} C = C$.

Polyhedral sets and polytopes

A half-space is a set of the form $H^-(u,\alpha) := \{x \in \mathbb{E} \mid \langle x, u\rangle \leqslant \alpha\}$; u is an outer normal to $H^-(u,\alpha)$, and is usually taken to be a unit vector. Half-spaces are clearly convex. The intersection of finitely many half-spaces is a polyhedral set Q, whose dimension is defined to be $\dim Q := \dim \operatorname{aff} Q$.

A bounded polyhedra set Q is also called a (convex) polytope. Equivalently

Theorem

Convex polytopes are the convex hulls of finite point-sets.

A simplex is the convex hull of an affinely independent set. A simple cone is a polyhedral set of the form $H^-(u_1,0) \cap \cdots \cap H^-(u_k,0)$, with $\{u_1,\ldots,u_k\}$ linearly independent.

Mappings

An affine mapping Φ preserves affine combinations; it is of the form $x\Phi = x\Psi + t$ for some linear mapping Ψ and translation vector t.

An isometry from one euclidean space to another is an affine mapping which preserves distance. More generally, a similarity is a mapping of the form $x \mapsto \alpha(x\Phi)$, where Φ is an isometry and $\alpha \neq 0$.

Orthogonal projection $\Pi_{\mathbb{L}} = \Pi$ on a linear subspace $\mathbb{L} \leqslant \mathbb{E}$ is given by

$$x\Pi = \sum_{i=1}^{k} \langle x, u_i \rangle u_i,$$

with $\{u_1, \ldots, u_k\}$ any orthonormal basis of $\mathbb{L}$.

For the future, we need

Theorem

Suppose that $\mathbb{L} \leqslant \mathbb{E}$, that $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{E}$, and that $a_j := e_j \Pi_{\mathbb{L}}$ for $j = 1, \ldots, n$. Then

$$\sum_{j=1}^{n} \|a_j\|^2 = \dim \mathbb{L}.$$

To see this, let $\{u_1, \ldots, u_k\}$ be any orthonormal basis of $\mathbb{L}$. Since $a_j = \sum_{i=1}^{k} \langle e_j, u_i \rangle u_i$ for each j, and $u_i = \sum_{j=1}^{n} \langle u_i, e_j \rangle e_j$ for each i, it follows that

$$\sum_{j=1}^{n} \|a_i\|^2 = \sum_{j=1}^{n} \sum_{i=1}^{k} \langle e_j, u_i \rangle^2 = \sum_{i=1}^{k} \|u_i\|^2 = k = \dim \mathbb{L},$$

as claimed.

Tensors

The tensor product $\mathbb{X} \otimes \mathbb{Y}$ of two vector spaces $\mathbb{X}, \mathbb{Y}$ is the universal vector space for bilinear mappings Φ on $\mathbb{X} \times \mathbb{Y}$, which satisfy

$$(\lambda_1 x_1 + \lambda_2 x_2, y)\Phi = \lambda_1 (x_1, y)\Phi + \lambda_2 (x_2, y)\Phi$$

for $x_1, x_2 \in \mathbb{X}$ and scalars λ_1, λ_2, and similarly for linearity in $\mathbb{Y}$. Note that $\dim(\mathbb{X} \otimes \mathbb{Y}) = \dim \mathbb{X} \cdot \dim \mathbb{Y}$.

The euclidean structure is induced on the tensor product of euclidean spaces by the inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle := \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle.$$

Finally, linear mappings Φ on $\mathbb{X}$ and Ψ on $\mathbb{Y}$ induce a linear mapping $\Phi \otimes \Psi$ on the tensor product $\mathbb{X} \otimes \mathbb{Y}$ by

$$(x \otimes y)(\Phi \otimes \Psi) := (x\Phi) \otimes (y\Psi).$$

NIP-sets

A nip-set (non-positive inner product set) is a set $U = \{u_1, u_2, \ldots\}$ of (unit) vectors such that $\langle u_j, u_k \rangle \leqslant 0$ for each $j \neq k$.

We say that $V = \{v_0, \ldots, v_m\}$ is a minimal positively dependent set if

$$\lambda_0 v_0 + \cdots + \lambda_m v_m = o$$

for some $\lambda_0, \ldots, \lambda_m > 0$, but each proper subset of V is linearly dependent.

Adding a suitable multiple of this relation to a general linear combination in $\operatorname{lin} V$ shows that

Theorem

A minimal positively dependent set positively spans its linear hull.

Theorem

A nip-set decomposes into mutually orthogonal minimal positively dependent sets and a linearly independent set.

For the proof, first suppose that V is linearly dependent, so that we can find $U := \{u_0, \ldots, u_k\} \subseteq V$ which is minimally linearly dependent. For $x = (\xi_0, \ldots, \xi_k) \in \mathbb{E}^{k+1}$ let $\Sigma(U, x) = \sum_{j=0}^{k} \xi_j u_j$, and define $|x| := (|\xi_0|, \ldots, |\xi_k|)$. Then $\Sigma(U, x) = o$ implies that

$$\begin{aligned} 0 = \left\| \Sigma(U, x) \right\|^2 &= \sum_{i,j=0}^{k} \xi_i \xi_j \langle u_i, u_j \rangle \\ &\geqslant \sum_{i,j=0}^{k} |\xi_i||\xi_j| \langle u_i, u_j \rangle = \left\| \Sigma(U, |x|) \right\|^2 \geqslant 0, \end{aligned}$$

whence $\Sigma(U, |x|) = o$. Thus U is minimally positively dependent.

With this same U, define $\mathbb{L} := \operatorname{lin} U$, and suppose that $v \in V \setminus U$. If $\langle x, v \rangle \neq 0$ for some $x \in \mathbb{L}$, then we can change the sign of x, if necessary, and assume that

$$\langle x, v \rangle > 0.$$

But $x \in \mathbb{L}$ implies that $x = \Sigma(U, z)$ for some $z = (\zeta_0, \ldots, \zeta_k)$ with $\zeta_j > 0$ for each $j = 0, \ldots, k$. Hence $\langle u_j, v \rangle > 0$ for at least one j, contradicting the assumption that V was a NIP-set.

The proof now proceeds by induction on the dimension, since clearly $\dim \operatorname{lin}(V \setminus U) < \dim V$. Eventually we conclude that V must be finite, and we end up with V decomposed into mutually orthogonal minimal positively dependent subsets, and possibly a residual linearly independent set. This completes the proof.

Fundamental regions

The mirrors (hyperplanes of reflexion) of a discrete group $\mathbf{G}$ generated by reflexions in hyperplanes acting on a euclidean space $\mathbb{E}$ divide the space into fundamental regions; these are polyhedral sets.

Remark

If, say, $\mathbf{H} = \langle S, T \rangle$, with two hyperplane reflexions S, T, then there are two possibilities. Either (as hyperplanes) S and T are parallel, in which case the outer normals are equal and opposite, or the fundamental regions of $\mathbf{H}$ are (simple) cones whose dihedral angles are some submultiple of π.

Theorem

If $\mathbf{G}$ is a discrete group generated by hyperplane reflexions, then the set U of unit outer normal vectors to the bounding hyperplanes of a fundamental region D of $\mathbf{G}$ is a nip-set.

Applying the remark to the subgroup **H** generated by any two reflexions in bounding hyperplanes of D shows that the dihedral angles of D are submultiples of π. Thus the outer normals do form a nip-set, as claimed.

Corollary

A fundamental region for a discrete group generated by hyperplane reflexions is a direct product of simplices and a simple cone.

We can confine our attention to irreducible groups; here, the fundamental region of **G** is the product of a simplex with a linear subspace, or a simple cone, the latter being the case when **G** is finite.

Coxeter diagrams

Let $U = \{u_1, \ldots, u_k\}$ be the set of unit outer normals to the fundamental region D of a discrete group **G** generated by hyperplane reflexions. With D we associate a Coxeter diagram $\mathcal{D}$, as follows. For each $j = 1, \ldots, k$, $\mathcal{D}$ has a node labelled j. The nodes i, j are joined by branch marked p_{ij} if $\langle u_i, u_j \rangle = -\cos(\pi/p_{ij})$, with the following exceptions:

- branches marked 2 are omitted – these correspond to reflexions which commute;
- labels 3 on branches are omitted, due to their frequency.

Thus connected Coxeter diagrams are precisely those of irreducible groups.

This leads very quickly to the familiar classification of the finite and discrete infinite hyperplane reflexion groups in euclidean spaces, in terms of their Coxeter diagrams. The key is that, if $\mathcal{D}$ is a Coxeter diagram of an irreducible infinite group, whose fundamental region is a simplex, then

- no mark on any branch of $\mathcal{D}$ can be increased,
- no new branch be added to $\mathcal{D}$.

Increasing a mark, leading from normal set U to V, say, would result in $\langle v_i, v_j \rangle < \langle u_i, u_j \rangle$ for some i, j, and hence in

$$0 = \|\Sigma(U, z)\|^2 > \|\Sigma(V, z)\|^2 \geqslant 0,$$

where $z > o$ is such that $\Sigma(U, z) = o$, a contradiction. Adjoining a new branch would violate the decomposition condition of a nip-set.

Example

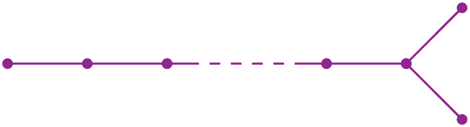
The unit vectors

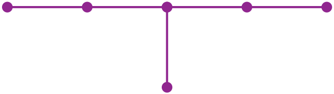
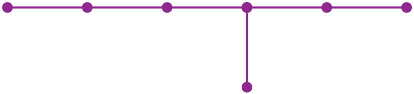
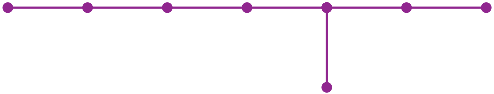
$$
\begin{aligned}
u_0 &:= -e_1, \\
u_j &:= \tfrac{1}{\sqrt{2}}(e_j - e_{j+1}) \qquad \text{for } j = 1, \ldots, d-1, \\
u_d &:= e_d,
\end{aligned}
$$

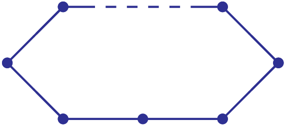
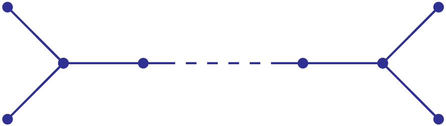
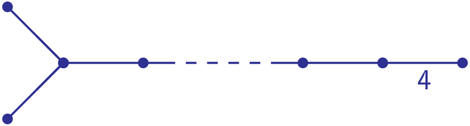
are minimally positively dependent; they satisfy

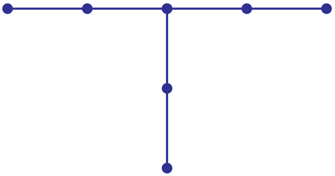
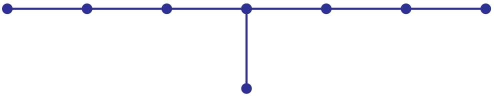
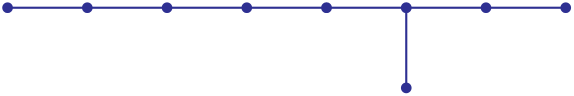
$$u_0 + \sqrt{2}(u_1 + \cdots + u_{d-1}) + u_d = o.$$

They determine a fundamental region for $\mathbf{R}_{d+1}$, the symmetry group of the d-dimensional cubic tiling, with Coxeter diagram

Notation	Diagram	Order
A_d		$(d+1)!$
B_d		$2^{d-1}d!$
C_d		$2^d d!$
D_2^p		$2p$
G_3		120
G_4		14400

Notation	Diagram	Order
$\mathbf{F}_4$		1152
$\mathbf{E}_6$		$72 \cdot 6!$
$\mathbf{E}_7$		$8 \cdot 9!$
$\mathbf{E}_8$		$192 \cdot 10!$

Notation	Diagram
P_{d+1}	
Q_{d+1}	
R_{d+1}	
S_{d+1}	

Notation	Diagram
T_7	
T_8	
T_9	

Notation	Diagram
U_5	
V_3	
W_2	

Products of reflexions

In the context of Petrie polygons, we need

Theorem

Let $\mathbf{G}$ *be an irreducible discrete group generated by hyperplane reflexions. If* $R_1, \ldots, R_k$ *are the reflexions in the bounding hyperplanes of a fundamental region of* $\mathbf{G}$*, then the fixed points of the product* $R_1 \cdots R_k$ *are just the points of* $R_1 \cap \cdots \cap R_k$ *(where reflexions are identified with their mirrors).*

The proof (which we shall not give) is by induction on k. Note that the order of the reflexions in the product is immaterial.

Rotation groups

Theorem

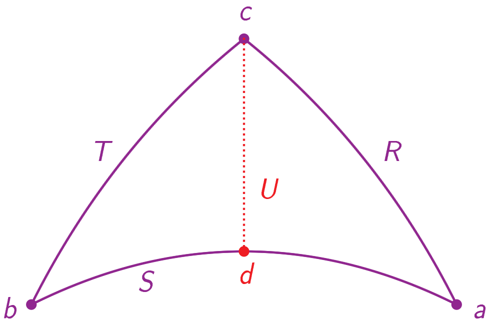
Each finite rotation group in $\mathbb{E}^3$ is a subgroup of index 2 in some reflexion group.

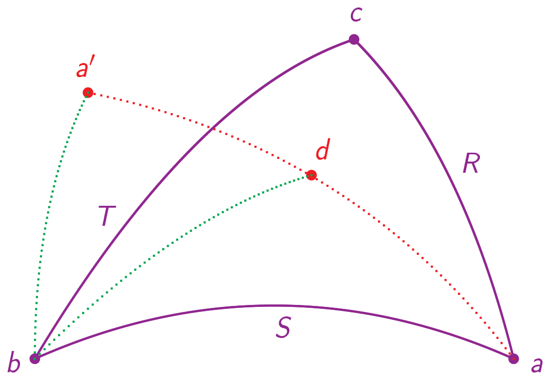
Corollary

Apart from cyclic groups, the finite rotation groups in $\mathbb{E}^3$ are $[2, q]^+$ (for $q \geqslant 2$), $[3, 3]^+$, $[3, 4]^+$ and $[3, 5]^+$.

The proof is on hand of the following diagrams. The key is that a rotation Φ can be expressed as a product $\Phi = ST$ of two planar reflexions, whose mirrors (for which we use the same notation) contain the axis L of Φ. Moreover, either S or T can be chosen freely.

$$\Phi = RS, \quad \Psi = ST$$





c
a′
d
R
T
S
b
a

Quaternions

The quaternions consist of all $\mathbf{x} = \xi_0 + \xi_1\mathbf{i} + \xi_2\mathbf{j} + \xi_3\mathbf{k}$, where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

They form an associative but not commutative division ring. The conjugate of $\mathbf{x}$ is $\bar{\mathbf{x}} := \xi_0 - \xi_1\mathbf{i} - \xi_2\mathbf{j} - \xi_3\mathbf{k}$; regarding $\mathbf{x}$ as a vector in $\mathbb{E}^4$, its norm is given by $\|\mathbf{x}\|^2 = \bar{\mathbf{x}}\mathbf{x} = \mathbf{x}\bar{\mathbf{x}}$. Note that $\overline{\mathbf{xy}} = \bar{\mathbf{y}}\,\bar{\mathbf{x}}$, so that $\|\mathbf{xy}\| = \|\mathbf{x}\|\|\mathbf{y}\|$. A unit quaternion $\mathbf{x}$ satisfies $\|\mathbf{x}\| = 1$.

The family $\mathbf{Q}$ of unit quaternions thus forms a group, with $\mathbf{a}^{-1} = \bar{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{Q}$. If $\mathbf{a} \in \mathbf{Q}$, then $\mathbf{x} \mapsto -\mathbf{a}\bar{\mathbf{x}}\mathbf{a}$ is the reflexion in the (linear) hyperplane with unit normal $\mathbf{a}$. As a consequence,

Theorem

The orthogonal group $\mathbf{O}_4$ consists of the mappings of the form $\mathbf{g}(\mathbf{a}, \mathbf{b})\colon \mathbf{x} \mapsto \bar{\mathbf{a}}\mathbf{x}\mathbf{b}$ and $\bar{\mathbf{g}}(\mathbf{a}, \mathbf{b})\colon \mathbf{x} \mapsto \bar{\mathbf{a}}\,\bar{\mathbf{x}}\mathbf{b}$. The mappings $\mathbf{g}(\mathbf{a}, \mathbf{b})$ form the special orthogonal group $\mathbf{SO}_4$.

Finite quaternion groups

The pure imaginary quaternions $\mathbf{x} = \xi_1 \mathbf{i} + \xi_2 \mathbf{j} + \xi_3 \mathbf{k}$ can be thought of as forming the subspace $\mathbb{E}^3$. A general $\mathbf{a} \in \mathbf{Q}$ can be written $\mathbf{a} = \cos\vartheta + \sin\vartheta \mathbf{u}$ for some ϑ, with $\mathbf{u} \in \mathbf{Q}$ pure imaginary. If, similarly, $\mathbf{b} = \cos\psi + \sin\psi \mathbf{v}$, then $\mathbf{g}(\mathbf{a}, \mathbf{b})$ is a double rotation in $\mathbb{E}^4$ through $\vartheta \pm \psi$.

Theorem

The mapping $\mathbf{g}(\mathbf{a}, \mathbf{a})$ induces a rotation in $\mathbb{E}^3$ through -2ϑ about the axis in direction $\mathbf{u}$. The induced homomorphism from $\mathbf{Q}$ to $\mathbf{SO}_3$ has kernel $\{\pm 1\}$.

Corollary

Apart from cyclic groups of odd order, each finite subgroup of $\mathbf{Q}$ is a 2-fold covering of a finite subgroup of $\mathbf{SO}_3$.

For brevity, we identify $\mathbf{x} \leftrightarrow (\xi_0, \xi_1, \xi_2, \xi_3)$. Ignoring the cyclic groups, for the finite groups coordinates can be chosen as follows.

Binary dihedral group $\widehat{\mathbf{D}}_n$ of order $4n$: all

$$(\cos(k\pi/n), \sin(k\pi/n), 0, 0), \quad (0, 0, \cos(k\pi/n), \sin(k\pi/n)).$$

Binary tetrahedral group $\widehat{\mathbf{A}}$ of order 24: all permutations of

$$(\pm 1, 0, 0, 0), \quad \tfrac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1).$$

Binary octahedral group $\widehat{\mathbf{C}}$ of order 48: $\widehat{\mathbf{A}}$ together with $\widehat{\mathbf{B}}$, all permutations of

$$\tfrac{1}{2}(\pm\sqrt{2}, \pm\sqrt{2}, 0, 0).$$

Binary icosahedral group $\widehat{\mathbf{G}}$ of order 120: $\widehat{\mathbf{A}}$ together with all even permutations of

$$\tfrac{1}{2}(\pm\tau, \pm 1, \pm\tau^{-1}, 0),$$

where $\tau = \frac{1}{2}(1 + \sqrt{5})$.

Remark
If we choose odd permutations instead, we get an isomorphic copy $\widehat{\mathbf{G}}^\dagger$ of $\widehat{\mathbf{G}}$, obtained by changing the sign of $\sqrt{5}$, or replacing τ by $-\tau^{-1}$.

Theorem
The isomorphic copy $\widehat{\mathbf{G}}^\dagger$ of $\widehat{\mathbf{G}}$ is conjugate to $\widehat{\mathbf{G}}$ under any element of $\widehat{\mathbf{B}}$.

Remark
We shall see that $\widehat{\mathbf{A}}$ can be regarded as the vertex-set of the regular 24-cell $\{3, 4, 3\}$, while $\widehat{\mathbf{G}}$ (or $\widehat{\mathbf{G}}^\dagger$) is that of the 120-cell $\{3, 3, 5\}$.

The subgroup $\widehat{\mathbf{V}} := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ of index 3 in $\widehat{\mathbf{A}}$ gives the vertex-set of the 4-dimensional cross-polytope $\{3, 3, 4\}$, while its complement $\widehat{\mathbf{A}} \setminus \widehat{\mathbf{V}}$ gives the 4-cube $\{4, 3, 3\}$.