Full rank:
non-classical polytopes

General considerations

We have seen that, if P is a d-dimensional faithful realization of a finite abstract regular d-polytope $\mathcal{P}$, with symmetry group $\mathbf{G} = \langle R_0, \ldots, R_{d-1} \rangle$, then (as mirrors)

$$\dim R_j = \begin{cases} j+1 \text{ or } d-1, & \text{if } j = 0, \ldots, d-3, \\ d-1, & \text{if } j = d-2 \text{ or } d-1. \end{cases}$$

Similarly, if P is a d-dimensional discrete faithful realization of an abstract regular $(d+1)$-apeirotope $\mathcal{P}$, with symmetry group $\mathbf{G} = \langle R_0, \ldots, R_d \rangle$, then

$$\dim R_j = \begin{cases} j \text{ or } d-1, & \text{if } j = 0, \ldots, d-2, \\ d-1, & \text{if } j = d-1 \text{ or } d. \end{cases}$$

Polytopes

First note that the operation $P \mapsto P^{\zeta} = P \otimes \{2\}$ always pairs up (finite) regular polytopes of full rank. This accounts for the change in dimension $d - 1 \leftrightarrow 1$ of the initial mirror R_0. Observe that, if $\dim R_0 = 1$, then $R_0^{\perp}$ is a hyperplane.

As we saw, in dimension 3, for each P we have $P^{\zeta} = Q^{\pi}$ for some (usually) different polyhedron Q.

In general, we have

Theorem

In dimension $d \geqslant 5$, for $j \geqslant 1$ only the case $\dim R_j = d - 1$ can occur.

The crucial case here is $j = 1$. The inductive proof uses the classification of the regular $(d - 1)$-polytopes of full rank; it can be assumed in the proof that $\dim R_j = d - 1$ for each $j \neq 1$.

For the d-simplex, $\{3^{d-1}\}^\zeta$ has $2(d+1)$ vertices (those of the simplex and their opposites), and symmetry group $\mathbf{S}_{d+1} \times \boldsymbol{C}_2$ of order $2(d+1)!$.

For the cross-polytope, $\{3^{d-2}, 4\}^\zeta$ has the same $2d$ vertices and the same symmetry group as $\{3^{d-2}, 4\}$. However, the facets $\{3^{d-2}\} \diamond \{2\} \cong \{3^{d-2}\} \# \{2\}$ contain all $2d$ vertices, so that the new polytope is flat (recall that this means that every vertex is incident with every facet).

What happens for the d-cube depends on whether d is even or odd. If d is even, then $\{4, 3^{d-2}\}^\zeta \cong \{4, 3^{d-2}\}$ is an isomorphic (but not congruent) copy. If d is odd, then $\{4, 3^{d-2}\}^\zeta \cong \boldsymbol{\{}4, 3^{d-2}\boldsymbol{\}}/2$, which is obtained from the cube by identifying opposite vertices. For example, $\{4, 3\}^\zeta = \{\frac{4}{1,2}, 3 : 3\}$, the Petrial of the tetrahedron.

The case $d = 4$ is only a little different. Here, the case $\dim R_1 = 2$ can arise from the Petrie operation π; however, this is only valid for the 4-cube $Q := \{4, 3, 3\}$ and Q^ζ. For the 4-cube,

$$P := \{4, 3, 3\}^\pi = \{\{4, \tfrac{4}{1,2} \mid 4\}, \{\tfrac{4}{1,2}, 3 : 3\}\},$$

a locally toroidal polytope. The facets are tori $\{4, \frac{4}{1,2} \mid 4\}$, which contain all 16 vertices of the cube; hence P is flat.

Further applying ζ to P merely produces an isomorphic, but not congruent, copy of P.

Remark

By halving, $\{3, 4, 3\}^\eta = \{4, 3, 3\}$, an interesting connexion which, nevertheless, yields nothing new. However, it implies that $[3, 3, 4]$ is a subgroup of $[3, 4, 3]$, in fact of index 3, and that the 16 vertices of $\{4, 3, 3\}$ occur among the 24 vertices of $\{3, 4, 3\}$.

Apeirotopes

We now deal with apeirotopes. If $\dim R_0 = 0$, then $P = Q^{\boldsymbol{\alpha}}$ is obtained from some rational regular d-polytope Q (of full rank) by the free abelian apeirotope construction $\boldsymbol{\alpha}$. In theory, such a $Q^{\boldsymbol{\alpha}}$ could be non-polytopal but, in practice, for full rank this does not occur.

In case $Q = \{3^{d-1}\}$ or $\{4, 3^{d-2}\}$, the apeirotopes $P = Q^{\boldsymbol{\alpha}}$ are universal as regular apeirotopes with facets in $\text{apeir}\{3^{d-2}\}$ or $\text{apeir}\{4, 3^{d-3}\}$, respectively, and vertex-figures Q.

On the other hand,

$$\{3^{d-2}, 4\}^{\boldsymbol{\alpha}} = \{4, 3^{d-2}, 4\}^{\boldsymbol{\kappa}_{02}},$$

with (invertible) $\boldsymbol{\kappa}_{02}$ replacing the initial hyperplane mirror R_0 by the mid-point of the initial edge of the cubic tiling $\{4, 3^{d-2}, 4\}$.

The operation κ applied to the classical regular honeycombs – that is, ζ applied to their vertex-figures – yields apeirotopes of much interest, particularly in dimension 4. Their facets will be regular apeirotopes of nearly full rank, so providing initial examples for the future.

In each dimension $d \geqslant 3$, we have $\{4, 3^{d-2}, 4\}^{\kappa}$. Since the new vertex-figure $\{3^{d-2}, 4\}^{\zeta}$ is flat, so is the apeirotope itself. We shall say more about the facets later; for now, we just remark that the 3-faces are Petrie-Coxeter sponges $\{4, \frac{6}{1,3} \mid 4\}$.

In a similar way, the 3-faces of $\{3, 3, 4, 3\}^{\kappa}$ and $\{3, 4, 3, 3\}^{\kappa}$ are Petrie-Coxeter sponges $\{6, \frac{6}{1,3} \mid 3\}$ and $\{6, \frac{4}{1,2} \mid 4\}$, respectively. It is striking that all three of these sponges occur as 3-faces.

It remains to note that $P := \{3,4,3,3\}^{\pi}$ does exist; its facets are locally toroidal apeirotopes $\{\{3,4\},\{4,\frac{4}{1,2} \mid 4\}\}$.

Because its Schläfli symbol is of the form $\{3,4,4,3\}$, it might be thought that P is self-dual. However, it is not, because the edge-figure $\{\frac{4}{1,2},3:3\} \cong \boldsymbol{\{}4,3\boldsymbol{\}}/2$ is not the dual of the 3-face $\{3,4\}$ (that is the cube $\{4,3\}$).

We can further apply $\boldsymbol{\kappa}$ to $P = \{3,4,3,3\}^{\pi}$, but the result is a little complicated to describe.

There is also an application of $\boldsymbol{\eta}$, but $\{3,3,4,3\}^{\boldsymbol{\eta}} = \{3,4,3,3\}$ is not new. However, it does show that $[3,3,4,3]$ is isomorphic to a proper subgroup of itself, actually of index 4. Further, the vertex-set of $\{3,3,4,3\}$ contains copies of that of its dual $\{3,4,3,3\}$.