Low dimensions

Dimensions $0, 1$

There is not too much to be said about the low-dimensional regular polytopes. In dimension 0, we just have the point-polytope. So far as realizations are concerned, we introduced the notation $\{1\}$ for the henogon, consisting of the single point $1 \in \mathbb{R}$.

In $\mathbb{R}$ itself, the only finite polytope is the (line) segment; again, the digon has the two vertices $\pm 1 \in \mathbb{R}$ when we discuss realizations. Its group is the cyclic group $\mathbf{C}_1 = \langle R_0 \rangle \cong \boldsymbol{C}_2$ where, for $\xi \in \mathbb{R}$,

$$\xi R_0 = -\xi.$$

The sole infinite example in $\mathbb{R}$ is the apeirogon $\{\infty\}$ $(= \{\frac{1}{0}\})$, whose vertex-set we can take to be $\mathbb{Z} \subset \mathbb{R}$. Its group is the infinite dihedral group $\mathbf{W}_2 = \langle R_0, R_1 \rangle$ where, for $\xi \in \mathbb{R}$,

$$\xi R_0 = 1 - \xi, \quad \xi R_1 = -\xi.$$

Polygons

In $\mathbb{E}^2$, things are more interesting. First, for each rational $p > 2$, we have the p-gon $\{p\}$, whose symmetry group $\langle R_0, R_1 \rangle$ is given (for example) by the matrices

$$R_0 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R_1 := \begin{bmatrix} \cos\frac{\pi}{p} & \sin\frac{\pi}{p} \\ \sin\frac{\pi}{p} & -\cos\frac{\pi}{p} \end{bmatrix}.$$

If $p = \frac{s}{t}$ in lowest terms, this is the (geometric) dihedral group $\mathbf{D}_2^s \cong \boldsymbol{D}_s$. With initial vertex $v = (1, 0)$, successive vertices of $\{p\}$ are $(\cos\frac{2k\pi}{p}, \sin\frac{2k\pi}{p})$ for $k = 0, 1, \ldots, s-1$.

Remark

We have already, in effect, dealt with all possible realizations of regular polygons, since they are either pure, or a blend with $\{2\}$ or $\{\infty\}$. In the latter case, we can have helical apeirogons with irrational turns; this is a rare occasion when they need to be mentioned.

Note particularly the zigzag apeirogon $\{\frac{2}{0,1}\} = \{2\} \# \{\infty\}$; up to similarity, we can take its vertices to consist of all $(k, (-1)^k \alpha)$ for some $\alpha > 0$.

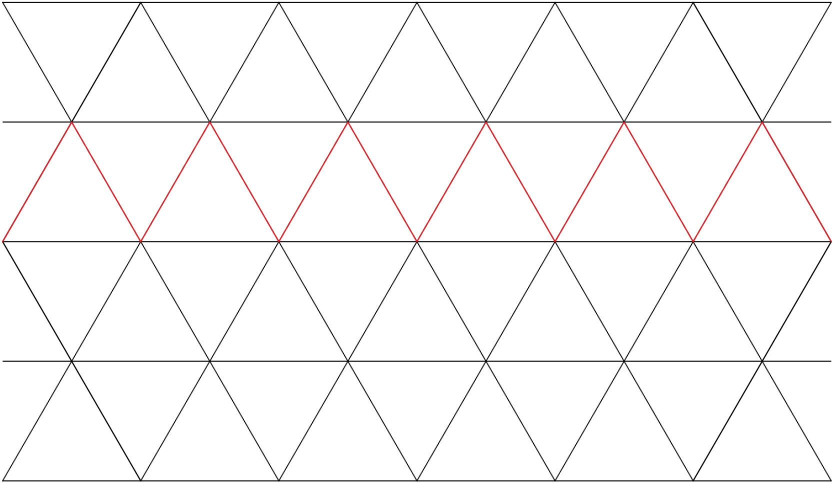
Apeirohedra

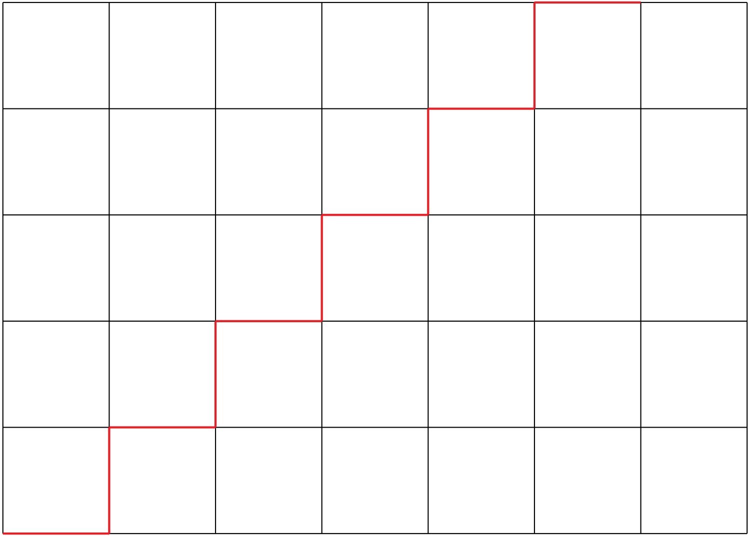
Since we have effectively dealt with lower ranks, we are left with the regular apeirohedra. Our assumption of discreteness restricts us to the three planar tessellations

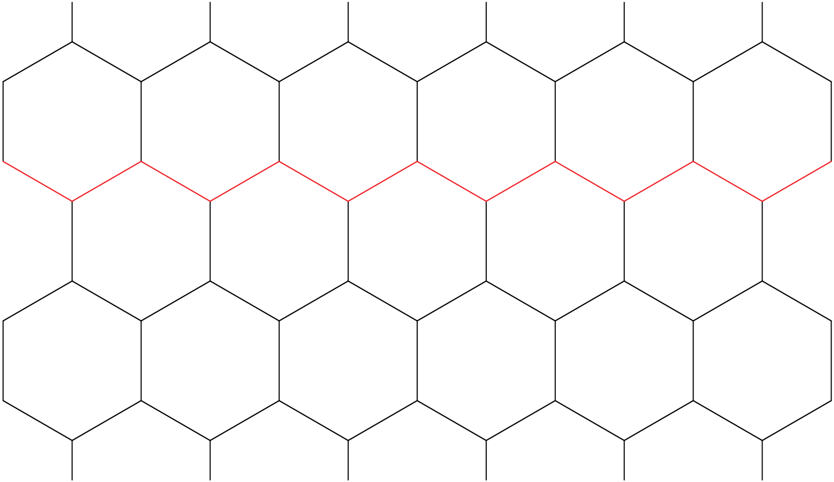
$$\{3,6\},\quad \{4,4\},\quad \{6,3\},$$

and their Petrials, which can also be represented by the free abelian apeirotope construction, because the initial reflexion R_0 here is the product of reflexions in two perpendicular lines, and so the reflexion in a point:

$$\{\tfrac{2}{0,1},6\} = \{3,6\}^{\pi} = \{6\}^{\alpha},$$
$$\{\tfrac{2}{0,1},4\} = \{4,4\}^{\pi} = \{4\}^{\alpha},$$
$$\{\tfrac{2}{0,1},3\} = \{6,3\}^{\pi} = \{3\}^{\alpha}.$$







The crystallographic regular polyhedra

Even though the order of the group of the tetrahedron $\{3,3\}$ and its Petrial is half that of the octahedron, nevertheless all the crystallographic 3-dimensional regular polyhedra fit into a single family.

$$\begin{array}{ccccc}
\{3,3\} & \overset{\pi}{\longleftrightarrow} & \{\frac{4}{1,2},3:3\} & & \\
\zeta\updownarrow & & \zeta\updownarrow & & \\
\{\frac{6}{1,3},3:4\} & \overset{\pi}{\longleftrightarrow} & \{4,3\} & & \\
 & & \delta\updownarrow & & \\
 & & \{3,4\} & \overset{\pi,\zeta}{\longleftrightarrow} & \{\frac{6}{1,3},4:3\}
\end{array}$$

Observe also the halving operation η: $\{4,3\} \to \{3,3\}$.

The non-crystallographic classical regular polyhedra

The classical regular polyhedra with the symmetry group of the icosahedron fit into a simple pattern, related by duality δ and the facetting operation φ_2.

$$
\begin{tikzcd}
\{5,3\} \arrow[r, leftrightarrow, "\delta"] & \{3,5\} \arrow[d, leftrightarrow, "\varphi_2"'] & & \\
& \{5,\frac{5}{2}\} \arrow[r, leftrightarrow, "\delta"] & \{\frac{5}{2},5\} \arrow[d, leftrightarrow, "\varphi_2"'] & \\
& & \{3,\frac{5}{2}\} \arrow[r, leftrightarrow, "\delta"] & \{\frac{5}{2},3\}
\end{tikzcd}
$$

This pattern is worth bearing in mind for the future.

The operations π and ζ both interchange the mirror vectors $(2,2,2)$ and $(1,2,2)$. However, on the pentagonal polytopes, they have different effects, and group the polyhedra in fours with the same vertex-figure.

$$\begin{tikzcd}
\{5,3\} \arrow[r, leftrightarrow, "\pi"] \arrow[d, leftrightarrow, "\zeta"'] & \{\frac{10}{1,5}, 3 : 5\} \arrow[d, leftrightarrow, "\zeta"'] \\
\{\frac{10}{3,5}, 3 : \frac{5}{2}\} \arrow[r, leftrightarrow, "\pi"] & \{\frac{5}{2}, 3\}
\end{tikzcd}$$

$$
\begin{tikzcd}
\{3,5\} \arrow[r, leftrightarrow, "\pi"] \arrow[d, leftrightarrow, "\zeta"'] & \{\frac{10}{1,5},5:3\} \arrow[d, leftrightarrow, "\zeta"'] \\
\{\frac{6}{1,3},5:\frac{5}{2}\} \arrow[r, leftrightarrow, "\pi"] & \{\frac{5}{2},5\}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\{3,\frac{5}{2}\} \arrow[r, leftrightarrow, "\pi"] \arrow[d, leftrightarrow, "\zeta"'] & \{\frac{10}{3,5},\frac{5}{2}:3\} \arrow[d, leftrightarrow, "\zeta"'] \\
\{\frac{6}{1,3},\frac{5}{2}:5\} \arrow[r, leftrightarrow, "\pi"] & \{5,\frac{5}{2}\}
\end{tikzcd}
$$

Of course, adding in duality ties all twelve into one family.

Blended apeirohedra

Each of the six regular apeirohedra in $\mathbb{E}^2$ can be blended with the digon $\{2\}$ or apeirogon $\{\infty\}$ to produce an apeirohedron in $\mathbb{E}^3$ of nearly full rank. These are paired by Petriality π, just as the planar ones are.

There is not much to say about these apeirohedra, except to note those which turn up as facets of regular apeirotopes of full rank.

$$\begin{array}{rcl}
\{3,3\}^{\alpha} & \text{has facet} & \{\frac{2}{0,1},3:6\}\,\#\,\{2\},\\
\{3,4\}^{\alpha} & \text{has facet} & \{\frac{2}{0,1},3:6\}\,\#\,\{2\},\\
\{4,3\}^{\alpha} & \text{has facet} & \{\frac{2}{0,1},4:4\}\,\#\,\{2\},\\
\{\frac{4}{1,2},3:3\}^{\alpha} & \text{has facet} & \{\frac{2}{0,1},4:4\}\,\#\,\{\infty\},\\
\{\frac{6}{1,3},4:3\}^{\alpha} & \text{has facet} & \{\frac{2}{0,1},6:3\}\,\#\,\{\infty\},\\
\{\frac{6}{1,3},3:4\}^{\alpha} & \text{has facet} & \{\frac{2}{0,1},6:3\}\,\#\,\{\infty\}.
\end{array}$$

Pure apeirohedra

The interest now lies in classifying the pure 3-dimensional regular apeirohedra.

We attack the problem using mirror vectors. Analysis of the various possibilities shows

Theorem

The mirror vectors of pure 3-dimensional regular apeirohedra are

$$(2,1,2), \quad (1,1,2), \quad (1,2,1), \quad (1,1,1).$$

Any others give finite polyhedra or blends, or are inconsistent.

Remark

The mirror vectors of the blended apeirohedra are sums of $(1,1,1)$ for the planar tessellations or $(0,1,1)$ for their Petrials, and $(0,1,1)$ for $\{2\}$ or $(0,0,1)$ for $\{\infty\}$.

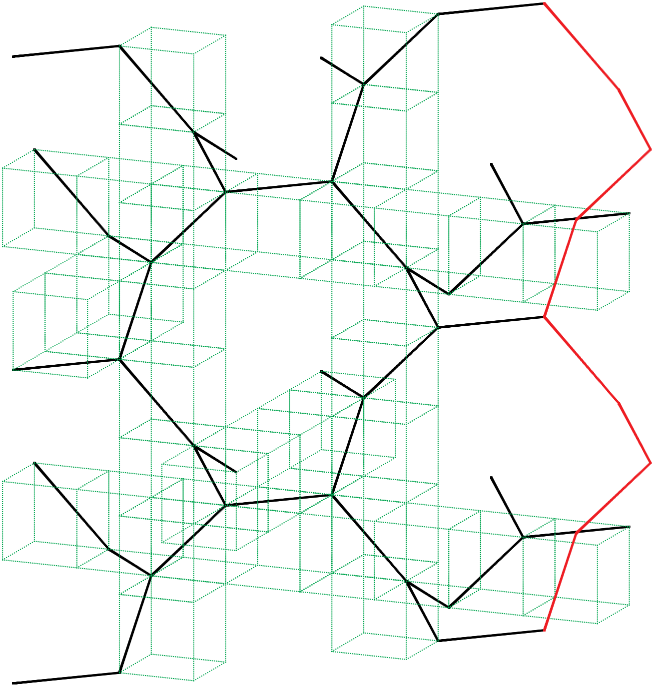
There is a nice trick to proceed from this point. Let the group of the pure regular apeirohedron P be $\mathbf{G} = \langle R_0, R_1, R_2 \rangle$, with initial vertex $o \in R_1 \cap R_2$. Let S_0 be the translate of R_0 through o and $S_j = R_j$ for $j = 1, 2$, so that $\langle S_0, S_1, S_2 \rangle$ is the special or point group of P. Let $T_j := S_j$ or $S_j^\perp$ as S_j is a plane or line, and $\mathbf{H} := \langle T_0, T_1, T_2 \rangle$. Then $\mathbf{H}$ is a crystallographic reflexion group, and so is $[3, 3]$, $[3, 4]$ or $[4, 3]$. We now reverse this process to find the $4 \cdot 3 = 12$ different apeirohedra.

We list all these apeirohedra, and the related finite polyhedra, according to their mirror vectors.

$(2,2,2)$	$\{3,3\}$	$\{3,4\}$	$\{4,3\}$
$(1,2,2)$	$\{\frac{4}{1,2},3:3\}$	$\{\frac{6}{1,3},4:3\}$	$\{\frac{6}{1,3},3:4\}$
$(2,1,2)$	$\{6,\frac{6}{1,3}\mid 3\}$	$\{6,\frac{4}{1,2}\mid 4\}$	$\{4,\frac{6}{1,3}\mid 4\}$
$(1,1,2)$	$\{\frac{4}{0,1},\frac{6}{1,3}:6\}$	$\{\frac{3}{0,1},\frac{4}{1,2}:6\}$	$\{\frac{3}{0,1},\frac{6}{1,3}:4\}$
$(1,2,1)$	$\{\frac{6}{1,3},6:\frac{4}{1,2}\}$	$\{\frac{6}{1,3},4:\frac{6}{1,3}\}$	$\{\frac{4}{1,2},6:\frac{6}{1,3}\}$
$(1,1,1)$	$\{\frac{3}{0,1},3:\frac{4}{0,1}\}$	$\{\frac{3}{0,1},4:\frac{3}{0,1}\}$	$\{\frac{4}{0,1},3:\frac{3}{0,1}\}$

Remark

The Schläfli symbols in the penultimate row do not provide complete descriptions.



Connexions

The pure apeirohedra are related in various ways, other than those indicated in the previous table.

First, apeirohedra in rows three and four are related by Petriality; in each of rows five and six we have a Petrie pair and a self-Petrie case.

As well as obvious duality δ and Petriality π, the apeirohedra with finite 2-faces are connected by halving η, while facetting provides further connexions:

$$\{4, \tfrac{6}{1,3} \mid 4\}^{\eta} = \{\tfrac{6}{1,3}, 6 : \tfrac{4}{1,2}\},$$

$$\{\tfrac{4}{1,2}, 6 : \tfrac{6}{1,3}\}^{\eta} = \{6, \tfrac{6}{1,3} \mid 3\},$$

$$\{\tfrac{6}{1,3}, 6 : \tfrac{4}{1,2}\}^{\varphi_2} = \{\tfrac{3}{0,1}, 3 : \tfrac{4}{0,1}\},$$

$$\{\tfrac{4}{1,2}, 6 : \tfrac{6}{1,3}\}^{\varphi_2} = \{\tfrac{4}{0,1}, 3 : \tfrac{3}{0,1}\}.$$

The Petrie-Coxeter apeirohedra arise in yet another way, through applications of κ to the crystallographic polyhedra. Since κ and π commute, we thus have

$$
\begin{aligned}
\{3,3\}^{\kappa} &= \{6, \tfrac{6}{1,3} \mid 3\},\\
\{3,4\}^{\kappa} &= \{6, \tfrac{4}{1,2} \mid 4\},\\
\{4,3\}^{\kappa} &= \{4, \tfrac{6}{1,3} \mid 4\},\\
\{\tfrac{4}{1,2}, 3 : 3\}^{\kappa} &= \{\tfrac{4}{0,1}, \tfrac{6}{1,3} : 6\},\\
\{\tfrac{6}{1,3}, 4 : 3\}^{\kappa} &= \{\tfrac{3}{0,1}, \tfrac{4}{1,2} : 6\},\\
\{\tfrac{6}{1,3}, 3 : 4\}^{\kappa} &= \{\tfrac{3}{0,1}, \tfrac{6}{1,3} : 4\}.
\end{aligned}
$$

Apeirotopes

There is just one classical regular 4-apeirotope in $\mathbb{E}^3$, namely, the familiar tiling of space by cubes $\{4,3,4\}$. However, $\mathbb{E}^3$ actually contains seven more regular 4-apeirotopes.

First, we can apply the Petrie operation π to the cubic tiling. This yields

$$\{4,3,4\} \quad \xleftrightarrow{\ \pi\ } \quad \{\{4, \tfrac{6}{1,3} \mid 4\}, \{\tfrac{6}{1,3}, 4 : 3\}\}.$$

Second, we can apply the free abelian apeirotope operation α to each of the six crystallographic regular polyhedra. Of course, the results will then be further related by operations such as π and κ (which is ζ applied to the vertex-figure).

As applied to the tetrahedron, cube and their Petrials, the results of α are not particularly interesting, except insofar as they begin sequences in all dimensions; we shall revisit this topic later.

For the octahedron, since we shall have the same vertex-figure as that of the cubic tiling, we might expect a deeper connexion. Indeed, the previous diagram can be expanded to

$$\begin{tikzcd} \{4,3,4\} \arrow[r, leftrightarrow, "\pi"] \arrow[d, leftrightarrow, "\kappa_{02}"'] & \{4, \frac{6}{1,3}, 4 : 3\}/\{13 \cdot 2; 3\} \arrow[d, leftrightarrow, "\kappa_{02}"'] \\ \{\frac{2}{0,1}, 3, 4 : \frac{3}{0,1}\} \arrow[r, leftrightarrow, "\pi"] & \{\frac{6}{1,3}, 4 : 3\}^{\alpha} \end{tikzcd}$$

Geometrically,

$$\kappa_{02} : (R_0, \ldots, R_3) \longleftrightarrow (R_0(R_2R_3)^2, R_1, R_2, R_3),$$

which just replaces the plane reflexion by the corresponding reflexion in the mid-point of the initial edge.

Finally, we have applications of Petrie contraction ϖ. Recall that, for $m \geqslant 3$, (abstractly) ϖ is the operation

$$\varpi\colon (\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{m-1}) \mapsto (\boldsymbol{r}_1, \boldsymbol{r}_0\boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_{m-1}) =: (\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{m-2}).$$

In $\mathbb{E}^3$, there are eight potential examples of regular 4-apeirotopes to which ϖ can be applied. However, six of them are free abelian apeirotopes Q^{α}, and for these we have $Q^{\alpha\varpi} = Q^{\kappa}$; the other two are the Petrie pair $\{4, 3, 4\}$ and $\{\{4, \frac{6}{1,3} \mid 4\}, \{\frac{6}{1,3}, 4 : 3\}\}$.

The basic example $\{4, 3, 4\}^{\varpi} = \{6, \frac{4}{1,2} \mid 4\}$ gives another way into the family derived from the Petrie-Coxeter sponges; indeed, this is other of the first two found by Petrie. Since the Petrie operation π commutes with ϖ, we also have

$$(\{\{4, \tfrac{6}{1,3} \mid 4\}, \{\tfrac{6}{1,3}, 4 : 3\}\})^{\varpi} = \{6, \tfrac{4}{1,2} \mid 4\}^{\pi} = \{\tfrac{3}{0,1}, \tfrac{4}{1,2} : 6\}.$$