Rigidity

Basic idea

We begin with an illustrative example.

Example

The small stellated dodecahedron has Schläfli symbol $\{\frac{5}{2}, 5\}$, which distinguishes it from the great dodecahedron with Schläfli symbol $\{5, \frac{5}{2}\}$. Both are isomorphic to the abstract polyhedron $\mathbf{\{}5, 5 \mid 3\mathbf{\}}$; however, the first has pentagrams as faces and pentagons as vertex-figures, while the second is the other way round.

More importantly, a regular polyhedron whose faces are pentagrams $\{\frac{5}{2}\}$ and vertex-figures are pentagons $\{5\}$ is a small stellated dodecahedron $\{\frac{5}{2}, 5\}$, uniquely up to similarity, rather than $\{5, \frac{5}{2}\}$. It is to be noted that there is no need to specify the hole $\{3\}$.

On the other hand, the general realization of $\mathbf{\{}5, 5 \mid 3\mathbf{\}}$ will be of the form $\{\frac{5}{1,2}, \frac{5}{1,2} \mid 3\}$, where now the hole must be given.

Fine Schläfli symbols

A fine Schläfli symbol determines a family of geometric regular polytopes by specifying in geometric terms the types of certain regular polygons which occur among its vertices. Recall here the fact that the vertices of sections can be asssumed to lie among the vertices of the polytope itself (though not usually with the induced symmetry).

The entries in these regular polygons will be generalized fractions. For instance, $\{\frac{6}{1,3}\} = \{6\} \# \{2\}$ is a skew hexagon inscribed in a hexagonal prism, $\{\frac{6}{2,3}\} = \{3\} \# \{2\}$ is a skew hexagon inscribed in a trigonal prism, while $\{\frac{3}{0,1}\} = \{\infty\} \# \{3\}$ is a 3-helix.

If a fine Schläfli symbol uniquely determines the shape, or similarity class, of a corresponding regular polytope P, then we call it rigid. More generally, we may ask what shapes of polytopes a fine Schläfli symbol does determine.

Remark

A complementary question asks whether a given geometric regular polytope has a fine Schläfli symbol which specifies its shape.

As an example, $\{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$ describes a regular apeirohedron whose faces and Petrie polygons are 4- and 3-helices, respectively. (The vertex-figure is a trigon, by which we mean a regular triangle.)

In fact, $\{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$ is rigid; we shall shortly see why. In this case, the fine Schl¨afli symbol gives no clue to the symmetry group; even knowing that the apeirohedron is rigid, it is not clear why its group is specified by the fact that the translations induced by a certain pair of face and Petrie polygon commute.

Realizations

In terms of realizations, a fine Schläfli symbol can be identified with a certain cone $\mathcal{P}$, say, of geometric regular polytopes. Then $\mathcal{P}$ being rigid means that it is 1-dimensional. Of course, $\mathcal{P}$ will be a subcone of the realization cone of some abstract regular polytope.

If two fine Schläfli symbols $\mathcal{P}, \mathcal{Q}$ are such that $\mathcal{Q}$ determines a subcone of $\mathcal{P}$, then we write $\mathcal{Q} \preccurlyeq \mathcal{P}$ or $\mathcal{P} \succcurlyeq \mathcal{Q}$. If they determine the same cone, so that $\mathcal{P} \preccurlyeq \mathcal{Q} \preccurlyeq \mathcal{P}$, then we write $\mathcal{P} \approx \mathcal{Q}$.

Example

Reverting to our earliest example, we can now write

$$\{5, \tfrac{5}{2}\} \preccurlyeq \boldsymbol{\{}5, 5 \mid 3\boldsymbol{\}}.$$

This emphasizes that fact that the fine Schläfli symbol often contains 'hidden' information.

Criteria for rigidity

Theorem

If the fine Schläfli type $\mathcal{P}$ has rank at least 2, planar 2-faces and rigid vertex-figure $\mathcal{Q}$, then $\mathcal{P}$ is rigid.

Corollary

The classical regular polytopes and honeycombs are rigid.

Theorem

If the fine Schläfli type $\mathcal{P}$ has rank at least 3, planar 2-faces, planar holes of its 3-faces, and vertex-figure of type $\mathcal{Q} \# \{2\}$ with $\mathcal{Q}$ rigid, then $\mathcal{P}$ is rigid.

Remark

As a generalization of the previous criterion, if the vertex-figure of a regular polyhedron P is a blend of at most k planar polygons (including the segment $\{2\}$), and k j-holes or zigzags for distinct j are specified planar polygons, then P will be rigid.

As an example, $\{3,3,3\}^{\varpi} = \{4, \frac{6}{2,3} \mid 3\} \cong \mathbf{\{}4, 6 \mid 3 : 5\mathbf{\}}$ is rigid. Abstractly, its Petrie polygon needs to be specified, but it is not needed for the fine Schläfli symbol (the geometric Petrie polygon is actually $\{\frac{5}{1,2}\}$). Now, only the vertex-figure distinguishes it from $\{4, \frac{6}{1,3} \mid 3\} \cong \mathbf{\{}4, 6 \mid 3\mathbf{\}}$ (also rigid, but with Petrie polygon $\{\frac{10}{1,3}\}$).

Indeed, for higher rank, one could have a vertex-figure which was a blend of rigid polytopes, rather than polygons. There is an obvious difficulty: how can one find a fine Schläfli symbol to pin down such a blend? This applies particularly to the second theorem.

Icosahedra

As an illustration, consider the icosahedron $\{3,5\}$. The (convex) icosahedron $\{3,5\}$ and great icosahedron $\{3,\frac{5}{2}\}$ are classical, and hence rigid.

The hemi-icosahedron (obtained by identifying opposite vertices, edges and faces of $\{3,5\}$) is

$$\{3,\tfrac{5}{1,2} : \tfrac{5}{1,2}\} \approx \{3,5:5\},$$

which is also rigid. The general regular icosahedron is a blend of these three and, of course, is not rigid.

Observe that we can write indifferently

$$\{3,\tfrac{5}{1,2} : \tfrac{5}{1,2}\} \preccurlyeq \{3,\tfrac{5}{1,2}\},$$
$$\{3,5:5\} \preccurlyeq \{3,5\}.$$

Dodecahedra

For the regular dodecahedron $\{5, 3\}$, the situation is rather different. Once again, we have the classical (convex) dodecahedron $\{5, 3\}$ and great stellated dodecahedron $\{\frac{5}{2}, 3\}$, which are rigid. There is a third pure faithful 4-dimensional realization of $\{5, 3\}$, whose fine Schläfli symbol $\{\frac{5}{1,2}, 3 : \frac{10}{1,3}\}$ is actually also rigid. But this is merely because there are no other pure faithful realizations, and – since the Petrie polygon is specified – $\{\frac{5}{1,2}, 3 : \frac{10}{1,3}\}$ cannot have $\{5, 3\}$ or $\{\frac{5}{2}, 3\}$ as a component.

However, all realizations of the hemi-dodecahedron $\{5, 3 : 5\}$ have fine Schläfli symbol $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$; in particular, there is no way that fine Schläfli symbols can distinguish the 4- and 5-dimensional pure realizations. (Certain regular hexagons in the edge-graph could make the distinction, but these cannot be specified in terms of the group generators.) Thus $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$ is, in general, 9-dimensional.

Polyhedra in $\mathbb{E}^3$

Example

As already observed, the classical regular polyhedra are all rigid, because they have planar faces and planar vertex-figures.

Example

The Petrials of the classical regular polyhedra are rigid. Here, it is the planar Petrie polygons and planar vertex-figures which ensure rigidity.

Remark

In a sense, the faces of these Petrials do not need to be specified. We could, for example, write

$$\{\cdot, 3 : 3\} = \{\tfrac{4}{1,2}, 3 : 3\} = \{3, 3\}^\pi.$$

Apeirohedra in $\mathbb{E}^3$

Example

The Petrie-Coxeter sponges, such as $\{4, \frac{6}{1,3} \mid 4\}$, are rigid; their faces and holes are planar polygons, and their vertex-figures are skew polygons (blends with two components).

Example

The Petrials of the Petrie-Coxeter sponges are rigid, without needing to specify the full combinatorial type. In fact, we have

$$\{\tfrac{4}{0,1}, \tfrac{6}{1,3} : 6\} \cong \{\infty, 6 : 6, 3\},$$
$$\{\tfrac{3}{0,1}, \tfrac{6}{1,3} : 4\} \cong \{\infty, 6 : 4, 4\},$$
$$\{\tfrac{3}{0,1}, \tfrac{4}{1,2} : 6\} \cong \{\infty, 4 : 6, 4\}.$$

Without going into too much detail, the reason is that the planar Petrie polygons determine the geometry of the helical faces (same angle at the vertices), and this is enough to fix their shapes. Observe, though, that we are now describing the apeirogonal faces by more than just the abstract symbol $\{\infty\}$.

An interesting consequence is that the Petrie-Coxeter sponges admit alternative descriptions, in terms of their Petrials:

$$\begin{aligned}
\{6, \tfrac{6}{1,3} : \tfrac{4}{0,1}\} &\cong \{6, 6 \mid 3\}, \\
\{4, \tfrac{6}{1,3} : \tfrac{3}{0,1}\} &\cong \{4, 6 \mid 4\}, \\
\{6, \tfrac{4}{1,2} : \tfrac{3}{0,1}\} &\cong \{6, 4 \mid 4\}.
\end{aligned}$$

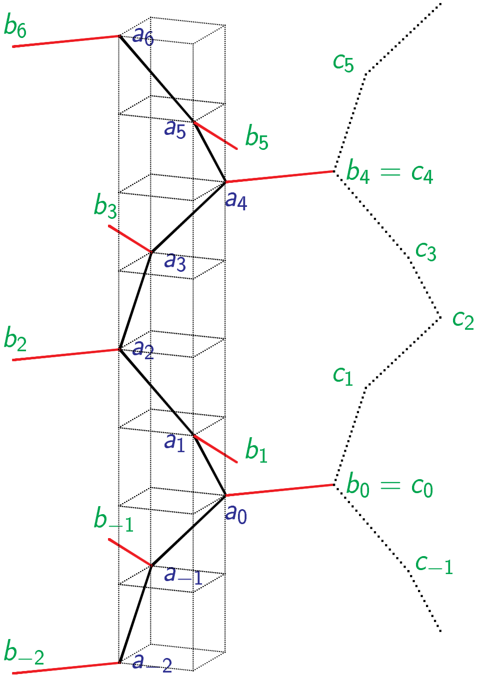
The geometric and abstract descriptions are now significantly different. Note also that we can write $\{6, \frac{6}{1,3} : \frac{4}{0,1}\} \approx \{6, \frac{6}{1,3} \mid 3\}$, and so on.

Rigidity of $\{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$

To prove the rigidity in this case, start with a 4-helical face $\ldots, a_{-1}, a_0, a_1, a_2, \ldots$. Through each a_j is a further edge $\{a_j, b_j\}$. The 3-helical Petrie apeirogons $\ldots, b_{j-2}, a_{j-2}, a_{j-1}, a_j, b_j, \ldots$, say, imply that we have parallelisms $a_{j-2} - b_{j-2} = b_j - a_j$, and so $b_{j+4} - a_{j+4} = b_j - a_j$ for each j.

Let $\ldots, c_{-1}, c_0, c_1, c_2, \ldots$ be the 4-helical face through $c_0 = b_0$ which does not contain the edge $\{a_0, b_0\}$ and is such that $\ldots, b_{-2}, a_{-2}, a_{-1}, a_0, b_0 = c_0, c_1, c_2, \ldots$ is a Petrie apeirogon. It follows that $c_j = a_{j-2} + (b_0 - a_{-2})$ for $j = 0, 1, 2$.

Using the Petrie apeirogon $\ldots, b_2, a_2, a_1, a_0, b_0 = c_0, c_{-1}, c_{-2}, \ldots$ similarly shows that $c_j = a_{j+2} + (b_0 - a_2)$ for $j = -2, -1, 0$. In view of the (local) translative symmetry $a_j \mapsto a_{j+4}$ of the initial helical face, we conclude that $c_j = a_{j-2} + (b_0 - a_{-2})$ for all j.



b_6
a_6
c_5
a_5
b_5
$b_4 = c_4$
a_4
b_3
c_3
a_3
c_2
b_2
a_2
c_1
a_1
b_1
$b_0 = c_0$
a_0
b_{-1}
c_{-1}
a_{-1}
b_{-2}
a_{-2}

We deduce that we have a family of 4-helical faces which are all translates of the initial one; adjacent members are connected by bridges like the $\{a_j, b_j\}$ with $c_j = b_j$, for $j \equiv 0 \pmod 4$.

There are two other such families, corresponding to the two other 4-helical faces which contain a_0. For these, the edges $\{a_0, a_{\pm 1}\}$ play the rôle of typical bridges to faces adjacent to initial 4-helical faces $\ldots, b_{\mp 1}, a_{\mp 1}, a_0, c_0, c_{\pm 1}, \ldots$.

A crucial implication is that the apeirohedron is 3-dimensional, and that the intrinsic translational symmetries of faces and Petrie apeirogons extend to global symmetries of the whole apeirohedron. The uniqueness of the shape of the apeirohedron is an immediate consequence.

Remark

It may be observed that, while we used the regularity of the faces and Petrie apeirogons, we did not appeal to the symmetries of the vertex-figure $\{3\}$.

Remark

We can apply the circuit criterion, showing that the basic edge-circuits are irregular decagons, one of which appears near the centre of the picture (we shall not go into the details). From one particular such decagon, we obtain the relator

$$\mathsf{JKJ}^{-1}\mathsf{K}^{-1}, \qquad \mathsf{J} = (01)^4,\ \mathsf{K} = (012)^3,$$

where inverting an index sequence reverses its order. This relator indeed corresponds to the fact that the translations associated with a certain pair of face and Petrie apeirogon commute.

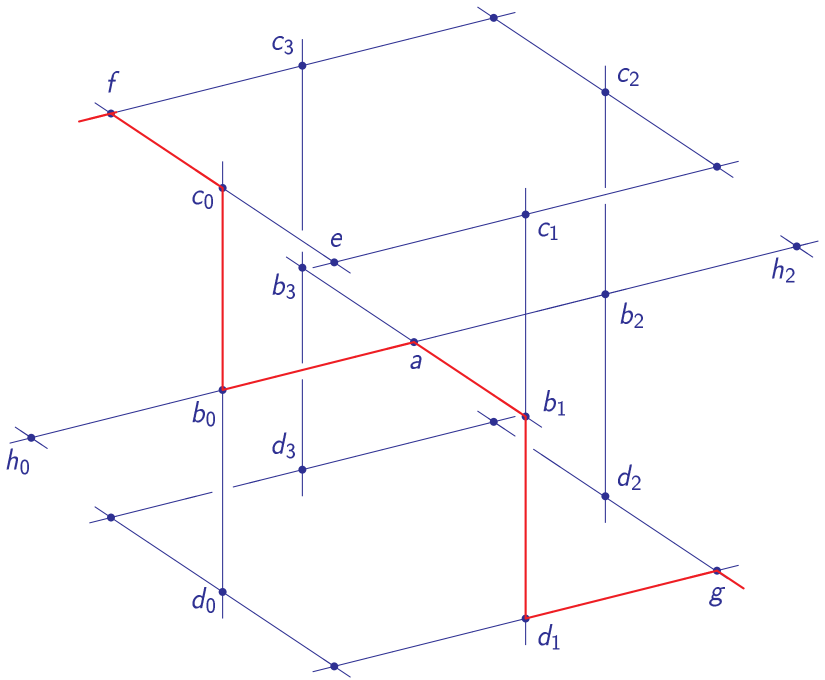
Rigidity of $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$

The regular apeirohedron $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$ is also rigid, but the proof here is a little different.

Remark

It is of interest that the regular apeirohedron $\{\frac{6}{1,3}, 4 : \frac{6}{1,3}\}$ has edge-graph isomorphic to that of $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$, but is not rigid. Indeed, since the faces and Petrie polygons are skew hexagons $\{\frac{6}{1,3}\}$, we can blend the apeirohedron with a segment $\{2\}$ to yield an isomorphic apeirohedron.

Indeed, it can be shown that general apeirohedron with fine Schläfli symbol $\{\frac{6}{1,3}, 4 : \frac{6}{1,3}\}$ is 4-dimensional.



c_3
f
c_2
c_0
e
c_1
b_3
h_2
b_2
a
b_0
b_1
h_0
d_3
d_2
d_0
g
d_1

From faces (or Petrie apeirogons) $\ldots, c_j, b_j, a, b_{j+1}, d_{j+1}, \ldots$ and $\ldots, d_j, b_j, a, b_{j+1}, c_{j+1}, \ldots$, we obtain

$$b_j - c_j = d_{j+1} - b_{j+1}, \quad b_j - d_j = c_{j+1} - b_{j+1}.$$

Since the vertex-figures $\{4\}$ are planar, the displacements

$$d(b_j) := b_j - \tfrac{1}{2}(c_j + d_j) = b_j - \tfrac{1}{2}(a + h_j)$$

satisfy

$$d(b_j) = -d(b_{j+1}) = d(b_{j+2}) = \cdots$$

But this first says that the holes $\ldots, h_j, b_j, a, b_{j+2}, h_{j+2}, \ldots$ are zigzags, with $d(a) = -d(b_j)$, and then that $d(a) = -d(a)$, or $d(a) = o$. Thus the holes are linear apeirogons $\{\frac{1}{0}\}$, and it easily follows that $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$ is 3-dimensional.