The geometry of regular polytopes

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The monograph *Abstract Regular Polytopes* (Cambridge University Press, 2002) by McMullen & Schulte – abbreviated below to 'ARP' – largely concentrated on the abstract theory of regular polytopes, although several parts of it did treat aspects of the geometric theory. There have been considerable developments in the abstract theory in recent years. However, this series of lectures will be more in the spirit of *Regular Polytopes* (3rd edition, Dover, 1973) by Coxeter, and will centre on the geometry of regular polytopes; a brief description of their proposed contents follows.

After a short introduction to the background of reflexion groups and the abstract theory, the first main topic will be *realizations*. Simply speaking, these attempt to describe geometric pictures of abstract polytopes in euclidean spaces, and various relationships among them. Much of the basic theory of realizations appeared in ARP, but the recent discovery of a product structure (for finite polytopes) has enabled many realization spaces to be determined which were formerly out of reach.

Another new notion is that of *rigidity* of geometric regular polytopes. As an example, the Schläfli symbol $\{5, \frac{5}{2}\}$ for the great dodecahedron – planar pentagonal faces fitting round each vertex in the manner of a pentagram – clearly distinguishes it from the small stellated dodecahedron $\{\frac{5}{2}, 5\}$, while neither symbol indicates the combinatorial structure of the isomorphic abstract polyhedron $\mathbf{\{}5, 5 \mid 3\mathbf{\}}$ (the entry 3 here is that of the *hole* – a triangle of edges which do not bound a face).

There are various operations which lead from one regular polytope to another, and constructions of new regular polytopes from (sometimes not regular) old ones. Many of them already occur in ARP, but these and some later ones will be discussed here.

The *rank* of an abstract regular polytope is a combinatorial equivalent of the dimension of a convex one. When rank and geometric dimension of a realization coincide, then it is of *full rank*. Among the striking advances since ARP is a complete description of the geometric regular polytopes of full rank, and also of those of *nearly* full rank (where dimension exceeds the rank by 1). While time will certainly not permit a complete account of all these, particularly in the case of nearly full rank, a survey will be given – with many examples – to treat the salient points.

The two monographs mentioned in the first paragraph provide useful background reading.