The complexity of positive first-order logic without equality

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Abstract

We study the complexity of evaluating positive equality-free sentences of first-order (FO) logic over a fixed, finite structure $\mathcal{B}$. This may be seen as a natural generalisation of the non-uniform quantified constraint satisfaction problem QCSP($\mathcal{B}$). *We introduce surjective hyper-endomorphisms and use them in proving a Galois connection that characterises definability in positive equality-free FO. Through an algebraic method, we derive a complete complexity classification for our problems as $\mathcal{B}$ ranges over structures of size at most three. Specifically, each problem is either in* L*, is* NP*-complete, is* co-NP*-complete or is* Pspace*-complete.*

1 Introduction

The evaluation problem under a logic $\mathcal{L}$ – here always a fragment of first-order logic (FO) – takes as input a structure (model) $\mathcal{B}$ and a sentence φ of $\mathcal{L}$, and asks whether $\mathcal{B} \models \varphi$.[1] When $\mathcal{L}$ is the *existential conjunctive positive* fragment of FO, $\{\exists, \wedge\}$-FO, the evaluation problem is equivalent to the much-studied *constraint satisfaction problem* (CSP). Similarly, when $\mathcal{L}$ is the *(quantified) conjunctive positive* fragment of FO, $\{\exists, \forall, \wedge\}$-FO, the evaluation problem is equivalent to the well-studied *quantified constraint satisfaction problem* (QCSP). In this manner, the QCSP is the generalisation of the CSP in which universal quantification is restored to the mix. In both cases it is essentially irrelevant whether or not equality is permitted in the sentences, as it may be propagated out by substitution. Much work has been done on the parameterisation of these problems by the structure $\mathcal{B}$ – that is, where $\mathcal{B}$ is fixed and only the sentence is input. It is conjectured [?] that the ensuing problems CSP($\mathcal{B}$) attain only the complexities P and NP-complete. This may appear surprising given that 1.) so many natural NP problems may be expressed as CSPs (see, e.g., myriad examples in [?]) and 2.) NP itself does not have this 'dichotomy' property (assuming P $\neq$ NP) [?]. While this *dichotomy conjecture* remains open, it has been proved for certain classes of $\mathcal{B}$ (e.g., for structures of size at most three [?] and for undirected graphs [?]) The like parameterisation of the QCSP is also well-studied, and while no overarching polychotomy has been conjectured, only the complexities P, NP-complete and Pspace-complete are known to be attainable (for trichotomy results on certain classes see [?, ?], as well as the dichotomy for boolean structures, e.g., in [?]).

In previous work, [?, ?], we have studied the evaluation problem, parameterised by the structure, under various fragments of FO obtained by restrictions on which of the symbols of $\{\exists, \forall, \wedge, \vee, \neg, =, \neq\}$ is permitted. Of course, many of the the ostensibly 2^7 such fragments may be discarded as totally trivial or as repetitions through de Morgan duality. There are four fragments each equivalent to the CSP and QCSP: these are $\{\exists, \wedge\}$-FO, $\{\exists, \wedge, =\}$-FO, $\{\forall, \vee\}$-FO, $\{\forall, \vee, \neq\}$-FO and $\{\exists, \forall, \wedge\}$-FO, $\{\exists, \forall, \wedge, =\}$-FO, $\{\exists, \forall, \vee\}$-FO, $\{\exists, \forall, \vee, \neq\}$-FO, respectively. Here, equivalent means that a complexity classification for one yields a complexity classification for the other; but, the complexity classes need not be the same. For example, the class of problems given by fixing the structure under $\{\exists, \wedge\}$-FO would display dichotomy between P and NP-complete iff the like class of problems under $\{\forall, \vee\}$-FO displays dichotomy between P and co-NP-complete. Various complexity classifications are obtained in [?, ?] and it is observed that the only interesting fragment, other than the eight associated with CSP and QCSP, is $\{\forall, \exists, \forall, \wedge\}$-FO.[2] The evaluation problem over $\{\exists, \forall, \wedge, \vee\}$-FO may be

[1] We resist the better known terminology of 'model checking problem' because in the majority of this paper we consider the structure $\mathcal{B}$ to be fixed.

[2] For many of the other fragments the complexity classification is nearly trivial. For example, this is true for $\{\exists, \wedge, \vee\}$-FO, $\{\forall, \wedge, \vee\}$-FO and $\{\exists, \forall, \wedge, \vee, \neg\}$-FO (also for these classes with $=$ or $\neq$). For others the

seen as the generalisation of the QCSP in which disjunction is returned to the mix. Note that the absence of equality is here important, as there is no general method for its being propagated out by substitution. Indeed, we will see that evaluating the related fragment $\{\exists, \forall, \wedge, \vee, =\}$-FO is Pspace-complete on any structure $\mathcal{B}$ of size at least two.

In this paper we initiate a study of the evaluation problem for the fragment $\{\exists, \forall, \wedge, \vee\}$-FO over a fixed relational $\mathcal{B}$ – the problem we denote $\{\exists, \forall, \wedge, \vee\}$-FO$(\mathcal{B})$. We demonstrate at least that this class displays a complexity-theoretic richness absent from those other fragments that are not associated with the CSP or QCSP. It is possibly to be hoped, however, that a full classification for this class is not as resistant as that for the CSP or QCSP. We undertake our study through the algebraic method that has been so fruitful in the study of the CSP and QCSP (see [**?**, **?**, **?**, **?**]). To this end, we define surjective hyper-endomorpisms and use them to define a new Galois connection that characterises definability under $\{\exists, \forall, \wedge, \vee\}$-FO.[3] We are able to prove a complete complexity classification for $\{\exists, \forall, \wedge, \vee\}$-FO$(\mathcal{B})$ when $\mathcal{B}$ ranges over structures of size at most three. On the class of boolean structures we see dichotomy between L and Pspace-complete. On the class of structures of size three we see tetrachotomy between L, NP-complete, co-NP-complete and Pspace-complete. Some of the results that appear in this paper had been obtained through adhoc methods in [**?**] – although there the tetrachotomy extends only to digraphs and not arbitrary relational structures. Also, little insight was provided as to the underlying properties of the classification. It is a pleasing consequence of our algebraic approach that we can give quite simple explanation to the delineation our subclasses.

The paper is organised as follows. In Section 2, we introduce the preliminaries, including the relevant Galois connection together with the central notions of surjective hyper-endomorphism (she) and down-she-monoid. In Section 3, we begin by outlining conditions under which the problem $\{\exists, \forall, \wedge, \vee\}$-FO$(\mathcal{B})$ either drops from or attains maximal complexity. We then proceed to classify the complexity of the problems $\{\exists, \forall, \wedge, \vee\}$-FO$(\mathcal{B})$, when $\mathcal{B}$ ranges over, firstly, boolean structures and, secondly, structures of size three. In the first instance a dichotomy – between L and Pspace-complete – is obtained; in the second instance a tetrachotomy – between L, NP-complete, co-NP-complete and Pspace-complete – is obtained. We conlcude, in Section 4, with some final remarks. Central to our tetrachotomy result is that the lattice of down-she-monoids in the three-element case has a certain structure. In the Appendix we prove that the lattice's structure is as we have claimed.

classification may be read through the Schaefer classification for boolean CSP and QCSP, because computational hardness is clear over fixed structures of size at least three. For example, this is the case for $\{\exists, \wedge, \neq\}$-FO, $\{\forall, \vee, =\}$-FO and $\{\exists, \forall, \wedge, \neq\}$-FO, $\{\exists, \forall, \vee, =\}$-FO. Note that the consideration of $\neq$ is not explicit in [**?**, **?**]. Similarly, fragments involving both quantifiers and $=$ or $\neq$ are not explicitly considered. In both cases, the results may be read off from de Morgan duality together with standard Schaefer class results (for which we refer to [**?**]).

[3] While this Galois connection appears here for the first time, it does follow a general recipe as outlined, e.g., in [**?**]. Note that it is not clear that the many different Galois connections associated with fragments of FO can be proved in a straightforwardly uniform manner.

2 Preliminaries

Throughout, let $\mathcal{B}$ be a finite structure, with domain B, over the finite relational signature σ. Let $\{\exists, \forall, \wedge, \vee\}$-FO and $\{\exists, \forall, \wedge, \vee, =\}$-FO be the positive fragments of first-order (FO) logic, without and with equality, respectively. An *extensional* relation is one that appears in the signature σ. We will usually denote extensional relations of $\mathcal{B}$ by R and other relations by S (or by some formula that defines them). In $\{\exists, \forall, \wedge, \vee\}$-FO the atomic formulae are exactly substitution instances of extensional relations. The problem $\{\exists, \forall, \wedge, \vee\}$-FO$(\mathcal{B})$ has:

- Input: a sentence $\varphi \in \{\exists, \forall, \wedge, \vee\}$-FO.
- Question: does $\mathcal{B} \models \varphi$?

The related problem $\{\exists, \forall, \wedge, \vee, =\}$-FO$(\mathcal{B})$ permits sentences φ that may involve equalities, in the obvious way. When $\mathcal{B}$ is of size one, the evaluation of any FO sentence may be accomplished in L (essentially, the quantifiers are irrelevant and the problem amounts to the *boolean sentence value problem*, see [**?**]). In this case, it follows that both $\{\exists, \forall, \wedge, \vee\}$-FO$(\mathcal{B})$ and $\{\exists, \forall, \wedge, \vee, =\}$-FO$(\mathcal{B})$ are also in L.

Consider the set B and its power set $\mathfrak{P}(B)$. A *hyper-operation* on B is a function $f : B \to \mathfrak{P}(B) \setminus \{\emptyset\}$ (that the image may not be the empty set corresponds to the hyper-operation being *total*, in the parlance of [**?**]). If the hyper-operation f has the additional property that

- for all $y \in B$, there exists $x \in B$ such that $y \in f(x)$,

then we designate (somewhat abusing terminology) f *surjective*. A surjective hyper-operation in which each element is mapped to a singleton set is identified with a *permutation* (bijection). A *surjective hyper-endomorphism* (she) of $\mathcal{B}$ is a surjective hyper-operation f on B that satisfies, for all extensional relations R of $\mathcal{B}$,

- if $R(x_1, \ldots, x_i) \in \mathcal{B}$ then, for all $y_1 \in f(x_1), \ldots, y_i \in f(x_i)$, $R(y_1, \ldots, y_i) \in \mathcal{B}$.

More generally, for $r_1, \ldots, r_k \in B$, we say f is *a she from* $(\mathcal{B}, r_1, \ldots, r_k)$ to $(\mathcal{B}, r'_1, \ldots, r'_k)$ if f is a she of $\mathcal{B}$ and $r'_1 \in f(r_1), \ldots, r'_k \in f(r_k)$. A she may be identified with a *surjective endomorphism* when each element is mapped to a singleton set. On finite structures surjective endomorphisms are necessarily automorphisms.

For $b_1, \ldots, b_{|B|}$ an enumeration of the elements of $\mathcal{B}$, let the quantifier-free formula $\Phi_{\mathcal{B}}(v_1, \ldots, v_{|B|})$ be a conjunction of the positive facts of $\mathcal{B}$, where the variables $v_1, \ldots, v_{|B|}$ correspond to the elements $b_1, \ldots, b_{|B|}$. That is, for R an extensional relation of $\mathcal{B}$, $R(v_{\lambda_1}, \ldots, v_{\lambda_i})$ appears as an atom in $\Phi_{\mathcal{B}}$ iff $R(b_{\lambda_1}, \ldots, b_{\lambda_i}) \in \mathcal{B}$. For example, let $\mathcal{K}_3$ be the antireflexive 3-clique, that is the structure with domain $\{0, 1, 2\}$ and single binary relation

$$E := \{(0,1), (1,0), (1,2), (2,1), (2,0), (0,2)\}.$$

Then

$$\begin{aligned}\Phi_{\mathcal{K}_3}(v_0, v_1, v_2) := \quad & E(v_0, v_1) \wedge E(v_1, v_0) \wedge E(v_1, v_2) \wedge \\ & E(v_2, v_1) \wedge E(v_2, v_0) \wedge E(v_0, v_2).\end{aligned}$$

The existential sentence $\exists v_1, \ldots, v_{|B|} \; \Phi_{\mathcal{B}}(v_1, \ldots, v_{|B|})$ is known as the *canonical query* of $\mathcal{B}$. More generally, for a (not necessarily distinct) l-tuple of elements $\mathbf{r} := (r_1, \ldots, r_l) \in B^l$, define the quantifier-free $\Phi_{\mathcal{B}(\mathbf{r})}(v_1, \ldots, v_l)$ to be the conjunction of the positive facts of $\mathbf{r}$, where the variables $v_1, \ldots, v_l$ correspond to the elements $r_1, \ldots, r_l$. That is, $R(v_{\lambda_1}, \ldots, v_{\lambda_i})$ appears as an atom in $\Phi_{\mathcal{B}(\mathbf{r})}$ iff $R(r_{\lambda_1}, \ldots, r_{\lambda_i}) \in \mathcal{B}$. For example,

$$\begin{aligned}\Phi_{\mathcal{K}_3(0,0,2)}(v_0, v_1, v_2) := \quad & E(v_0, v_2) \wedge E(v_2, v_0) \wedge \\ & E(v_1, v_2) \wedge E(v_2, v_1).\end{aligned}$$

We refer to elements in $\mathcal{B}$ as r, s, t (also x, y), or $b_1, \ldots, b_{|B|}$ when this is an enumeration. We reserve u, v, w to refer to variables in FO formulae.

2.1 Galois Connections

For a set F of surjective hyper-operations on the finite domain B, let $\mathsf{Inv}(F)$ be the set of relations on B of which each $f \in F$ is a she (when these relations are viewed as a structure over B). We say that $S \in \mathsf{Inv}(F)$ is invariant or *preserved* by (the hyper-operations in) F. Let $\mathsf{shE}(\mathcal{B})$ be the set of shes of $\mathcal{B}$. Let $\mathsf{Aut}(\mathcal{B})$ be the set of automorphisms of $\mathcal{B}$.

Let $\langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-FO}}$ and $\langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee,=\}\text{-FO}}$ be the sets of relations that may be defined on $\mathcal{B}$ in $\{\exists,\forall,\wedge,\vee\}$-FO and $\{\exists,\forall,\wedge,\vee,=\}$-FO, respectively.

Lemma 1. *Let $\mathbf{r} := (r_1, \ldots, r_k)$ be a k-tuple of elements of $\mathcal{B}$. There exists:*

(i). *a formula $\theta_{\mathbf{r}}(u_1, \ldots, u_k) \in \{\exists,\forall,\wedge,\vee,=\}$-FO s.t. $(\mathcal{B}, r'_1, \ldots, r'_k) \models \theta_{\mathbf{r}}(u_1, \ldots, u_k)$ iff there is an automorphism from $(\mathcal{B}, r_1, \ldots, r_k)$ to $(\mathcal{B}, r'_1, \ldots, r'_k)$.*

(ii). *a formula $\theta_{\mathbf{r}}(u_1, \ldots, u_k) \in \{\exists,\forall,\wedge,\vee\}$-FO s.t. $(\mathcal{B}, r'_1, \ldots, r'_k) \models \theta_{\mathbf{r}}(u_1, \ldots, u_k)$ iff there is a she from $(\mathcal{B}, r_1, \ldots, r_k)$ to $(\mathcal{B}, r'_1, \ldots, r'_k)$.*

Proof. For Part (i), let $b_1, \ldots, b_{|B|}$ an enumeration of the elements of $\mathcal{B}$ and $\Phi_{\mathcal{B}}(v_1, \ldots, v_{|B|})$ be the associated conjunction of positive facts. Set $\theta_{\mathbf{r}}(u_1, \ldots, u_k) :=$

$$\begin{aligned}\exists v_1, \ldots, v_{|B|} \quad & \Phi_{\mathcal{B}}(v_1, \ldots, v_{|B|}) \wedge \\ & \forall v \; (v = v_1 \vee \ldots \vee v = v_{|B|}) \wedge \\ & u_1 = v_{\lambda_1} \wedge \ldots \wedge u_k = v_{\lambda_k},\end{aligned}$$

where $r_1 = b_{\lambda_1}, \ldots, r_k = b_{\lambda_k}$.

[Part (ii).] This will require greater dexterity. Let $\mathbf{r} \in B^k$, $\mathbf{s} := (b_1, \ldots, b_{|B|})$ be an enumeration of B and $\mathbf{t} \in B^{|B|}$. Recall that $\Phi_{\mathcal{B}(\mathbf{r},\mathbf{s})}(u_1, \ldots, u_k, v_1, \ldots, v_{|B|})$ is a conjunction of the positive facts of $(\mathbf{r}, \mathbf{s})$, where the variables $(\mathbf{u}, \mathbf{v})$ correspond to the elements $(\mathbf{r}, \mathbf{s})$. Similarly, $\Phi_{\mathcal{B}(\mathbf{r},\mathbf{s},\mathbf{t})}(u_1, \ldots, u_k, v_1, \ldots, v_{|B|}, w_1, \ldots, w_{|B|})$ is the conjunction of the positive facts of $(\mathbf{r}, \mathbf{s}, \mathbf{t})$, where the variables $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ correspond to the elements $(\mathbf{r}, \mathbf{s}, \mathbf{t})$. Set $\theta_{\mathbf{r}}(u_1, \ldots, u_k) :=$

$$\exists v_1, \ldots, v_{|B|} \; \Phi_{\mathcal{B}(\mathbf{r},\mathbf{s})}(u_1, \ldots, u_k, v_1, \ldots, v_{|B|}) \wedge \forall w_1 \ldots w_{|B|}$$

$$\bigvee_{\mathbf{t} \in B^{|B|}} \Phi_{\mathcal{B}(\mathbf{r},\mathbf{s},\mathbf{t})}(u_1, \ldots, u_k, v_1, \ldots, v_{|B|}, w_1, \ldots, w_{|B|}).$$

[Part (ii), backwards.] Suppose f is a she from $(\mathcal{B}, r_1, \ldots, r_k)$ to $(\mathcal{B}', r'_1, \ldots, r'_k)$, where $\mathcal{B}' := \mathcal{B}$ (we will wish to differentiate the two occurrences of $\mathcal{B}$). We aim to prove that $\mathcal{B}' \models \theta_{\mathbf{r}}(r'_1, \ldots, r'_k)$. Choose arbitrary $s'_1 \in f(b_1), \ldots, s'_{|B|} \in f(b_{|B|})$ as witnesses for $v_1, \ldots, v_{|B|}$. Let $\mathbf{t}' := (t'_1, \ldots, t'_{|B|}) \in B'^{|B|}$ be any valuation of $w_1, \ldots, w_{|B|}$ and take arbitrary $t_1, \ldots, t_{|B|}$ s.t. $t'_1 \in f(t_1), \ldots, t'_{|B|} \in f(t_{|B|})$ (here we use surjectivity). Let $\mathbf{t} := (t_1, \ldots, t_{|B|})$. It follows from the definition of she that

$$\begin{aligned}\mathcal{B}' \models \quad & \Phi_{\mathcal{B}(\mathbf{r},\mathbf{s})}(r'_1, \ldots, r'_k, s'_1, \ldots, s'_{|B|}) \wedge \\ & \Phi_{\mathcal{B}(\mathbf{r},\mathbf{s},\mathbf{t})}(r'_1, \ldots, r'_k, s'_1, \ldots, s'_{|B|}, t'_1, \ldots, t'_{|B|}).\end{aligned}$$

[Part (ii), forwards.] Assume that $\mathcal{B}' \models \theta_{\mathbf{r}}(r'_1, \ldots, r'_k)$, where $\mathcal{B}' := \mathcal{B}$. Let $b'_1, \ldots, b'_{|B|}$ be an enumeration of $B' := B$.[4] Choose some witness elements $s'_1, \ldots, s'_{|B|}$ for $v_1, \ldots, v_{|B|}$ and a witness tuple $\mathbf{t} := (t_1, \ldots, t_{|B|}) \in B^{|B|}$ s.t.

$$\begin{aligned}(\dagger) \quad \mathcal{B}' \models \quad & \Phi_{\mathcal{B}(\mathbf{r},\mathbf{s})}(r'_1, \ldots, r'_k, s'_1, \ldots, s'_{|B|}) \wedge \\ & \Phi_{\mathcal{B}(\mathbf{r},\mathbf{s},\mathbf{t})}(r'_1, \ldots, r'_k, s'_1, \ldots, s'_{|B|}, b'_1, \ldots, b'_{|B|}).\end{aligned}$$

Consider the following partial hyper-operations from $B \to \mathfrak{P}(B') \setminus \{\emptyset\}$.

1. $f_{\mathbf{r}}$ given by (the union of) $r_1 \mapsto \{r'_1\}, \ldots, r_k \mapsto \{r'_k\}$.

[4] One may imagine $b_1, \ldots, b_{|B|}$ and $b'_1, \ldots, b'_{|B|}$ to be the same enumeration, but this is not essential. In any case, we will wish to keep the dashes on the latter set to remind us they are in $\mathcal{B}'$ and not $\mathcal{B}$.

2. $f_{\mathbf{s}}$ given by $f_{\mathbf{s}}(b_i) = \{s'_i\}$.

3. $f_{\mathbf{t}}$ given by $b'_i \in f_{\mathbf{t}}(b_j)$ iff $t_i = b_j$.

Let $f := f_{\mathbf{r}} \cup f_{\mathbf{s}} \cup f_{\mathbf{t}}$; f is a hyper-operation whose surjectivity is guaranteed by $f_{\mathbf{t}}$ (note that totality is guaranteed by $f_{\mathbf{s}}$). That f is a she follows from the right-hand conjunct of (†). $\square$

Theorem 1. *For a finite structure $\mathcal{B}$ we have*

(i). $\langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee,=\}\text{-}\mathsf{FO}} = \mathsf{Inv}(\mathsf{Aut}(\mathcal{B}))$ *and*

(ii). $\langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}} = \mathsf{Inv}(\mathsf{shE}(\mathcal{B}))$.

Proof. Part (i) is well-known and may be proved in a similar, albeit simpler, manner to Part (ii), which we now prove.

$[\varphi(\mathbf{v}) \in \langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}} \Rightarrow \varphi(\mathbf{v}) \in \mathsf{Inv}(\mathsf{shE}(\mathcal{B})).]$ This is proved by induction on the complexity of $\varphi(\mathbf{v})$.[5]

(Base Case.) $\varphi(\mathbf{v}) := R(\mathbf{v})$.[6] Follows from the definition of she.

(Inductive Step.) There are four subcases. We progress through them in a workmanlike fashion. Take $f \in \mathsf{shE}(\mathcal{B})$.

$\varphi(\mathbf{v}) := \psi(\mathbf{v}) \wedge \psi'(\mathbf{v})$. Let $\mathbf{v} := (v_1, \ldots, v_l)$. Suppose $\mathcal{B} \models \varphi(x_1, \ldots, x_l)$, then both $\mathcal{B} \models \psi(x_1, \ldots, x_l)$ and $\mathcal{B} \models \psi(x_1, \ldots, x_l)$. By Inductive Hypothesis (IH), for any $y_1 \in f(x_1), \ldots, y_l \in f(x_l)$, both $\mathcal{B} \models \psi(y_1, \ldots, y_l)$ and $\mathcal{B} \models \psi(y_1, \ldots, y_l)$, whence $\mathcal{B} \models \varphi(y_1, \ldots, y_l)$.

$\varphi(\mathbf{v}) := \psi(\mathbf{v}) \vee \psi'(\mathbf{v})$. Let $\mathbf{v} := (v_1, \ldots, v_l)$. Suppose $\mathcal{B} \models \varphi(x_1, \ldots, x_l)$, then one of $\mathcal{B} \models \psi(x_1, \ldots, x_l)$ or $\mathcal{B} \models \psi(x_1, \ldots, x_l)$; w.l.o.g. the former. By IH, for any $y_1 \in f(x_1), \ldots, y_l \in f(x_l)$, $\mathcal{B} \models \psi(y_1, \ldots, y_l)$, whence $\mathcal{B} \models \varphi(y_1, \ldots, y_l)$.

$\varphi(\mathbf{v}) := \forall w \; \psi(\mathbf{v}, w)$. Let $\mathbf{v} := (v_1, \ldots, v_l)$. Suppose $\mathcal{B} \models \forall w \; \psi(x_1, \ldots, x_l, w)$, then for each x', $\mathcal{B} \models \psi(x_1, \ldots, x_l, x')$. By IH, for any $y_1 \in f(x_1), \ldots, y_l \in f(x_l)$, we have for all y' (remember f is surjective), $\mathcal{B} \models \psi(y_1, \ldots, y_l, y')$, whereupon $\mathcal{B} \models \forall w \; \psi(y_1, \ldots, y_l, w)$.

$\varphi(\mathbf{v}) := \exists w \; \psi(\mathbf{v}, w)$. Let $\mathbf{v} := (v_1, \ldots, v_l)$. Suppose $\mathcal{B} \models \exists w \; \psi(x_1, \ldots, x_l, w)$, then for some x', $\mathcal{B} \models \psi(x_1, \ldots, x_l, x')$. By IH, for any $y_1 \in f(x_1), \ldots, y_l \in f(x_l), y' \in f(x')$ (remember $f(x')$ can not be empty), $\mathcal{B} \models \psi(y_1, \ldots, y_l, y')$, whereupon $\mathcal{B} \models \exists w \; \psi(y_1, \ldots, y_l, w)$.

$[S \in \mathsf{Inv}(\mathsf{shE}(\mathcal{B})) \Rightarrow S \in \langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}}.]$ Consider the k-ary relation $S \in \mathsf{Inv}(\mathsf{shE}(\mathcal{B}))$. Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ be the tuples of S. Set

$$\theta_S(u_1, \ldots, u_k) := \theta_{\mathbf{r}_1}(u_1, \ldots, u_k) \vee \ldots \vee \theta_{\mathbf{r}_m}(u_1, \ldots, u_k).$$

Manifestly, $\theta_S(u_1, \ldots, u_k) \in \{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}$. That $\theta_S(u_1, \ldots, u_k) = S$ now follows from Part (ii) of the previous lemma since $S \in \mathsf{Inv}(\mathsf{shE}(\mathcal{B}))$. $\square$

[5]Throughout the paper, the presence of, e.g., $\mathbf{v}$ in $\psi(\mathbf{v})$ should not be taken as indication that *all* $\mathbf{v}$ appear free in ψ.

[6]The variables $\mathbf{v}$ may appear multiply in R and in any order. Thus R is an instance of an extensional relation under substitution and permutation of positions.

In the following, $\leq_{\mathsf{L}}$ indicates the existence of a logspace many-to-one reduction.

Theorem 2. *Let $\mathcal{B}$ and $\mathcal{B}'$ be finite structures over the same domain B.*

(i). *If* $\mathsf{Aut}(\mathcal{B}) \subseteq \mathsf{Aut}(\mathcal{B}')$ *then* $\{\exists,\forall,\wedge,\vee,=\}\text{-}\mathsf{FO}(\mathcal{B}') \leq_{\mathsf{L}} \{\exists,\forall,\wedge,\vee,=\}\text{-}\mathsf{FO}(\mathcal{B})$.

(ii). *If* $\mathsf{shE}(\mathcal{B}) \subseteq \mathsf{shE}(\mathcal{B}')$ *then* $\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}(\mathcal{B}') \leq_{\mathsf{L}} \{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}(\mathcal{B})$.

Proof. Again, Part (i) is well-known and the proof is similar to that of Part (ii), which we give. If $\mathsf{shE}(\mathcal{B}) \subseteq \mathsf{shE}(\mathcal{B}')$, then $\mathsf{Inv}(\mathsf{shE}(\mathcal{B}')) \subseteq \mathsf{Inv}(\mathsf{shE}(\mathcal{B}))$. From the previous theorem, it follows that $\langle \mathcal{B}' \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}} \subseteq \langle \mathcal{B} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}}$. Recalling that $\mathcal{B}'$ contains only a finite number of extensional relations, we may therefore effect a Logspace reduction from $\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}(\mathcal{B}')$ to $\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}(\mathcal{B})$ by straightforward substitution of predicates. $\square$

2.2 Down-she-monoids

Consider a finite domain B. The *identity* hyper-operation id_B is defined by $x \mapsto \{x\}$. Given hyper-operations f and g, define the *composition* $g \circ f$ by $x \mapsto \{z : \exists y \; z \in g(y) \wedge y \in f(x)\}$. Finally, a hyper-operation f is a *sub-hyper-operation* of g – denoted $f \subseteq g$ – if $f(x) \subseteq g(x)$, for all x. A set of surjective hyper-operations on a finite set B is a *down-she-monoid*, if it contains id_B, and is closed under composition and sub-hyper-operations (of course, not all sub-hyper-operations of a surjective hyper-operation are surjective – we are only concerned with those that are). id_B is a she of all structures, and, if f and g are shes of $\mathcal{B}$, then so is $g \circ f$. Further, if g is a she of $\mathcal{B}$, then so is f for all $f \subseteq g$. It follows that $\mathsf{shE}(\mathcal{B})$ is always a down-she-monoid. The down-she-monoids of B form a lattice under (set-theoretic) inclusion and, as per the Galois connection of the previous section, classify the complexities of $\{\exists,\forall,\wedge,\vee\}\text{-}\mathsf{FO}(\mathcal{B})$. If F is a set of surjective hyper-operations on B, then let $\langle F \rangle$ denote the minimal down-she-monoid containing the operations of F. If F is the singleton $\{f\}$, then, by abuse of notation, we write $\langle f \rangle$ instead of $\langle \{f\} \rangle$

For a surjective hyper-operation f, define its inverse f^{-1} by $x \mapsto \{y : f(y) = x\}$. Note that f^{-1} is also a surjective hyper-operation and $(f^{-1})^{-1} = f$, though $f \circ f^{-1} = id_B$ only if f is a permutation. For a set of surjective hyper-operations F, let $F^{-1} := \{f^{-1} : f \in F\}$. If F is a down-she-monoid then so is F^{-1}. We will see this algebraic duality resonates with the de Morgan duality of $\exists$ and $\forall$, and the complexity-theoretic duality of NP and co-NP. However, we resist discussing it further as it plays no direct role in the derivation of our results.

A *permutation subgroup* on a finite set B is a set of permutations of B closed under composition. It may easily be verified that such a set contains the identity and is closed under inverse. A permutation subgroup may be identified with a particular type of down-she-monoid in which all hyper-operations have only singleton sets in their range. The permutation subgroups form a lattice under inclusion whose minimal element contains just the identity and whose maximal element is the symmetric group $S_{|B|}$. As per the Galois connection of the previous section, this lattice classifies the complexities of $\{\exists, \forall, \wedge, \vee, =\}$-$\mathsf{FO}(\mathcal{B})$ – although we shall see these are relatively uninteresting.

In the lattice of down-she-monoids, the minimal element still contains just id_B, but the maximial element contains all surjective hyper-operations. However, the lattice of permutation subgroups always appears as a sub-lattice within the lattice of down-she-monoids.

3 Classification results

We are now in a position to study the interplay between the shes of a structure $\mathcal{B}$ and the complexity of the problem $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B}$.

3.1 Shes inducing lower complexity

We begin by studying three classes of shes, the presence of which reduces the complexity of the problem $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$. Let $\mathcal{B}$ be a finite structure, with distinct elements b, b'. We define the following shes.

- $\forall_b : B \to \mathfrak{P}(B) \setminus \{\emptyset\}$, where $\forall_b(x) := B$, if $x = b$, and $\forall_b(x) := \{x\}$, otherwise.
- $\exists_b : B \to \mathfrak{P}(B) \setminus \{\emptyset\}$, where $\exists_b(x) := \{x, b\}$.
- $\forall_b\exists_{b'} : B \to \mathfrak{P}(B) \setminus \{\emptyset\}$, where $\forall_b\exists_{b'}(x) := B$, if $x = b$, and $\forall_b\exists_{b'}(x) := \{b'\}$, otherwise.

We call their classes $\forall$-, $\exists$- and $\forall\exists$-hyper-operations, respectively. In Figure 3.1, four digraphs $\mathcal{G}_1$–$\mathcal{G}_4$ are drawn. For typographic reasons we will mark-up, e.g., the surjective hyper-operation $0 \mapsto \{0, 1\}$, $1 \mapsto \{1\}$ and $2 \mapsto \{1, 2\}$ as $\begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}$. It may easily be verified that the down-she monoids $\mathsf{shE}(\mathcal{G}_1)$–$\mathsf{shE}(\mathcal{G}_4)$ are as follows.

$$\begin{array}{cccc} \mathsf{shE}(\mathcal{G}_1) & \mathsf{shE}(\mathcal{G}_2) & \mathsf{shE}(\mathcal{G}_3) & \mathsf{shE}(\mathcal{G}_4) \\ \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\rangle & \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 012 \\ \hline 2 & 2 \end{array} \right\rangle & \left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \\ \hline 2 & 012 \end{array} \right\rangle & \left\langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 1 \\ \hline 2 & 012 \end{array} \right\rangle \end{array}$$

Figure 1. Sample digraphs admitting $\forall$-, $\exists$- and $\forall\exists$-hyper-operations as shes.

We see that $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ admit the shes $\exists_1$, $\forall_1$ and $\forall_2\exists_1$, respectively. $\mathcal{G}_4$ admits each of the shes $\forall_0$, $\forall_2$, $\exists_1$, $\forall_0\exists_1$ and $\forall_2\exists_1$.

Remarks 1. We have not considered shes $\forall_b\exists_b$, defined as before but with $b' := b$. The down-she-monoid $\langle\forall_b\exists_b\rangle$ is easily seen to contain all surjective hyper-operations. It follows that any structure $\mathcal{B}$ that has $\forall_b\exists_b$ as a she already has all shes of the form $\forall_{b'}\exists_{b''}$ with $b' \neq b''$.

Note that the down-she-monoids $\langle\forall_b\exists_{b'}\rangle$ and $\langle\{\forall_b, \exists_{b'}\}\rangle = \langle\forall_b \circ \exists_{b'}\rangle$ do not in general coincide, though the first is always a subset of the following two. Also, we note the identities $\exists_b^{-1} = \forall_b$, $\forall_b^{-1} = \exists_b$ and $(\forall_b\exists_{b'})^{-1} = \forall_{b'}\exists_b$.

We now give a series of three lemmas, one associated with each of the shes $\forall_b$, $\exists_b$ and $\forall_b\exists_{b'}$. They will ultimately be used in a form of quantifier elimination that will diminish the complexity of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$, if $\mathcal{B}$ has one of these shes.

Lemma 2. *Let $\varphi(u, \mathbf{v})$ be a formula of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}$. Let $\mathcal{B}$ be a finite structure with $\forall_b$ as a she. Then*

$$\mathcal{B} \models \forall u\, \varphi(u, \mathbf{v}) \iff \mathcal{B} \models \varphi(b, \mathbf{v}).$$

Proof. We proceed by induction on the complexity of the formula φ. In each case, the forward direction ($\Rightarrow$) is trivial; we prove the backward.

(Base case.) $\varphi(u, \mathbf{v}) := R(u, \mathbf{v})$. That $R(b, \mathbf{v})$ implies $\forall u\, R(u, \mathbf{v})$ follows immediately from the she $\forall_b$.

(Inductive Step.) There are four subcases.

$\varphi(u, \mathbf{v}) := \psi(u, \mathbf{v}) \wedge \psi'(u, \mathbf{v})$. Assume $\psi(b, \mathbf{v}) \wedge \psi'(b, \mathbf{v})$ to derive both $\psi(b, \mathbf{v})$ and $\psi'(b, \mathbf{v})$. By IH, derive both $\forall u\, \psi(u, \mathbf{v})$ and $\forall u\, \psi'(u, \mathbf{v})$, which yields $\forall u\, \psi(u, \mathbf{v}) \wedge \psi'(u, \mathbf{v})$.

$\varphi(u, \mathbf{v}) := \psi(u, \mathbf{v}) \vee \psi'(u, \mathbf{v})$. Assume $\psi(b, \mathbf{v}) \vee \psi'(b, \mathbf{v})$ to derive, w.l.o.g. $\psi(b, \mathbf{v})$. By IH, derive $\forall u\, \psi(u, \mathbf{v})$, which yields the weaker $\forall u\, \psi(u, \mathbf{v}) \vee \psi'(u, \mathbf{v})$.

$\varphi(u, \mathbf{v}) := \forall w\ \psi(u, \mathbf{v}, w)$. Assume $\forall w\ \psi(b, \mathbf{v}, w)$ to derive, by IH, $\forall w \forall u\, \psi(u, \mathbf{v}, w)$, which is $\forall u \forall w\ \psi(u, \mathbf{v}, w)$.

$\varphi(u, \mathbf{v}) := \exists w\ \psi(u, \mathbf{v}, w)$. Assume $\exists w\ \psi(b, \mathbf{v}, w)$ to derive, by IH, $\exists w \forall u\ \psi(u, \mathbf{v}, w)$, which yields the weaker $\forall u \exists w\ \psi(u, \mathbf{v}, w)$. $\square$

The previous lemma has the following, dual version.

Lemma 3. *Let $\varphi(u, \mathbf{v})$ be a formula of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}$. Let $\mathcal{B}$ be a finite structure with $\exists_b$ as a she. Then*

$$\mathcal{B} \models \exists u\, \varphi(u, \mathbf{v}) \iff \mathcal{B} \models \varphi(b, \mathbf{v}).$$

Proof. We proceed by induction on the complexity of the formula φ. In each case, the backward direction ($\Leftarrow$) is trivial; we prove the forward.

(Base case.) $\varphi(u, \mathbf{v}) := R(u, \mathbf{v})$. That $\exists u\, R(u, \mathbf{v})$ implies $R(b, \mathbf{v})$ follows immediately from the she $\exists_b$.

(Inductive Step.) There are four subcases.

$\varphi(u, \mathbf{v}) := \psi(u, \mathbf{v}) \wedge \psi'(u, \mathbf{v})$. Assume $\exists u\ \psi(u, \mathbf{v}) \wedge \psi'(u, \mathbf{v})$ to derive the weaker $\exists u\ \psi(u, \mathbf{v})$ and $\exists u\ \psi'(u, \mathbf{v})$. By IH, derive both $\psi(b, \mathbf{v})$ and $\psi'(b, \mathbf{v})$, which yields $\psi(b, \mathbf{v}) \wedge \psi'(b, \mathbf{v})$.

$\varphi(u, \mathbf{v}) := \psi(u, \mathbf{v}) \vee \psi'(u, \mathbf{v})$. Assume $\exists u\ \psi(u, \mathbf{v}) \vee \psi'(u, \mathbf{v})$ to derive, w.l.o.g. $\exists u\ \psi(u, \mathbf{v})$. By IH, derive $\psi(b, \mathbf{v})$, which yields the weaker $\psi(b, \mathbf{v}) \vee \psi'(b, \mathbf{v})$.

$\varphi(u, \mathbf{v}) := \forall w\ \psi(u, \mathbf{v}, w)$. Assume $\exists u \forall w\ \psi(u, \mathbf{v}, w)$ to derive the weaker $\forall w \exists u\ \psi(u, \mathbf{v}, w)$, and then, by IH, $\forall w\ \psi(b, \mathbf{v}, w)$.

$\varphi(u, \mathbf{v}) := \exists w\ \psi(u, \mathbf{v}, w)$. Assume $\exists u \exists w\ \psi(u, \mathbf{v}, w)$, which is $\exists w \exists u\ \psi(u, \mathbf{v}, w)$, to derive, by IH, $\exists w\ \psi(b, \mathbf{v}, w)$. □

Lemma 4 (Interpolation). *Let $\varphi(\mathbf{u}, \mathbf{v})$ be a formula of $\{\exists, \forall, \wedge, \vee\}$-FO where $\mathbf{u}$ and $\mathbf{v}$ are of arities j and l, respectively. Let $\mathcal{B}$ be a finite structure with $\forall_b \exists_{b'}$ as a she. For all $c_1, \ldots, c_j \in B \setminus \{b, b'\}$, and for all $\mathbf{v} := (v_1, \ldots, v_l) \in \{b, b'\}^l$,*

$$\begin{array}{rl} \mathcal{B} \models \varphi(b, \ldots, b, \mathbf{v}) & \stackrel{(I)}{\Longrightarrow} \mathcal{B} \models \varphi(c_1, \ldots, c_j, \mathbf{v}) \\ & \stackrel{(II)}{\Longrightarrow} \mathcal{B} \models \varphi(b', \ldots, b', \mathbf{v}), \end{array}$$

where $b, \ldots, b$ and $b', \ldots, b'$ signify j-tuples of bs and b's, respectively.

Proof. We prove (I) and (II) simultaneously by induction on the complexity of the formula φ.

(Base case.) $\varphi(\mathbf{u}, \mathbf{v}) := R(\mathbf{u}, \mathbf{v})$. That $R(b, \ldots, b, \mathbf{v})$ implies $R(c_1, \ldots, c_j, \mathbf{v})$ implies $R(b', \ldots, b', \mathbf{v})$ follows immediately from the she $\forall \exists_{b,b'}$.

(Inductive Step.) There are four subcases.

$\varphi(\mathbf{u}, \mathbf{v}) := \psi(\mathbf{u}, \mathbf{v}) \wedge \psi'(\mathbf{u}, \mathbf{v})$. For (I), assume $\psi(b, \ldots, b, \mathbf{v}) \wedge \psi'(b, \ldots, b, \mathbf{v})$ to derive both $\psi(b, \ldots, b, \mathbf{v})$ and $\psi'(b, \ldots, b, \mathbf{v})$, whence both $\psi(c_1, \ldots, c_j, \mathbf{v})$ and $\psi'(c_1, \ldots, c_j, \mathbf{v})$, and, therefore, $\psi(c_1, \ldots, c_j, \mathbf{v}) \wedge \psi'(c_1, \ldots, c_j, \mathbf{v})$. Case (II) follows in similar fashion.

$\varphi(\mathbf{u}, \mathbf{v}) := \psi(\mathbf{u}, \mathbf{v}) \vee \psi'(\mathbf{u}, \mathbf{v})$. For (I), assume $\psi(b, \ldots, b, \mathbf{v}) \vee \psi'(b, \ldots, b, \mathbf{v})$ to derive w.l.o.g. $\psi(b, \ldots, b, \mathbf{v})$, whence $\psi(c_1, \ldots, c_j, \mathbf{v})$ and then the weaker $\psi(c_1, \ldots, c_j, \mathbf{v}) \vee \psi'(c_1, \ldots, c_j, \mathbf{v})$. Case (II) follows in the like fashion.

$\varphi(\mathbf{u}, \mathbf{v}) := \forall w\ \psi(\mathbf{u}, \mathbf{v}, w)$. For (I), assume $\forall w\ \psi(b, \ldots, b, \mathbf{v}, w)$, which implies both $\psi(b, \ldots, b, \mathbf{v}, b)$ and $\psi(b, \ldots, b, \mathbf{v}, b')$. By IH, we can derive $\psi(c_1, \ldots, c_j, \mathbf{v}, b)$ from the former and $\psi(c_1, \ldots, c_j, \mathbf{v}, b')$ from the latter. But we can also derive $\psi(c_1, \ldots, c_j, \mathbf{v}, c)$, for any $c \in B \setminus \{b, b'\}$, from $\psi(b, \ldots, b, \mathbf{v}, b)$ by considering the '$\mathbf{u}$' of the IH to be a $j+1$-tuple incorporating the final b. This proves $\forall w\ \psi(c_1, \ldots, c_j, \mathbf{v}, w)$. Diagrammatically:

$$\begin{array}{ccc} & & \psi(c_1, \ldots, c_j, \mathbf{v}, c) \\ & \nearrow & \\ \psi(b, \ldots, b, \mathbf{v}, b) & \rightarrow & \psi(c_1, \ldots, c_j, \mathbf{v}, b) \\ \psi(b, \ldots, b, \mathbf{v}, b') & \rightarrow & \psi(c_1, \ldots, c_j, \mathbf{v}, b') \end{array}$$

For (II), assume $\forall w\ \psi(c_1, \ldots, c_j, \mathbf{v}, w)$, which implies both $\psi(c_1, \ldots, c_j, \mathbf{v}, b)$ and $\psi(c_1, \ldots, c_j, \mathbf{v}, b')$. By IH, we can derive $\psi(b', \ldots, b', \mathbf{v}, b)$ from the former and $\psi(b', \ldots, b', \mathbf{v}, b')$ from the latter. But we can now also derive $\psi(b', \ldots, b', \mathbf{v}, c)$ from $\psi(b', \ldots, b', \mathbf{v}, b)$, for any $c \in B \setminus \{b, b'\}$, by the IH on (I). This proves $\forall w\ \psi(b', \ldots, b', \mathbf{v}, w)$.

$$\begin{array}{ccc} & & \psi(b', \ldots, b', \mathbf{v}, c) \\ & & \uparrow \\ \psi(c_1, \ldots, c_j, \mathbf{v}, b) & \rightarrow & \psi(b', \ldots, b', \mathbf{v}, b) \\ \psi(c_1, \ldots, c_j, \mathbf{v}, b') & \rightarrow & \psi(b', \ldots, b', \mathbf{v}, b') \end{array}$$

$\varphi(\mathbf{u}, \mathbf{v}) := \exists w\ \psi(\mathbf{u}, \mathbf{v}, w)$. For (I), assume $\exists w\ \psi(b, \ldots, b, \mathbf{v}, w)$. This gives either $\psi(b, \ldots, b, \mathbf{v}, b)$, $\psi(b, \ldots, b, \mathbf{v}, b')$ or $\psi(b, \ldots, b, \mathbf{v}, c)$, for some $c \in B \setminus \{b, b'\}$. The first two cases yield $\psi(c_1, \ldots, c_j, \mathbf{v}, b)$ and $\psi(c_1, \ldots, c_j, \mathbf{v}, b')$, respectively, by IH. However, $\psi(b, \ldots, b, \mathbf{v}, c)$ implies $\psi(b, \ldots, b, \mathbf{v}, b')$, by IH on (II) and this a fortiori gives $\psi(c_1, \ldots, c_j, \mathbf{v}, b')$, by the IH. Having derived $\exists w\ \psi(c_1, \ldots, c_j, \mathbf{v}, w)$, we are done.

$$\begin{array}{ccc} \psi(b, \ldots, b, \mathbf{v}, b) & \rightarrow & \psi(c_1, \ldots, c_j, \mathbf{v}, b) \\ \psi(b, \ldots, b, \mathbf{v}, b') & \rightarrow & \psi(c_1, \ldots, c_j, \mathbf{v}, b') \\ \uparrow & & \\ \psi(b, \ldots, b, \mathbf{v}, c) & & \end{array}$$

For (II), assume $\exists w\ \psi(c_1, \ldots, c_j, \mathbf{v}, w)$. This gives either $\psi(c_1, \ldots, c_j, \mathbf{v}, b)$, $\psi(c_1, \ldots, c_j, \mathbf{v}, b')$ or $\psi(c_1, \ldots, c_j, \mathbf{v}, c)$, for some $c \in B \setminus \{b, b'\}$. The first two cases yield $\psi(b', \ldots, b', \mathbf{v}, b)$ and $\psi(b', \ldots, b', \mathbf{v}, b')$, respectively, by IH. The last case also gives $\psi(b', \ldots, b', \mathbf{v}, b')$, by IH, considering '$\mathbf{u}$' to be a $j+1$-tuple incorporating the final c.

$$\begin{array}{ccc} \psi(c_1, \ldots, c_j, \mathbf{v}, b) & \rightarrow & \psi(b', \ldots, b', \mathbf{v}, b) \\ \psi(c_1, \ldots, c_j, \mathbf{v}, b') & \rightarrow & \psi(b', \ldots, b', \mathbf{v}, b') \\ & \nearrow & \\ \psi(c_1, \ldots, c_j, \mathbf{v}, c) & & \end{array}$$

□

Corollary 1. *Let $\varphi(u, \mathbf{v})$ be a formula of $\{\exists, \forall, \wedge, \vee\}$-FO where $\mathbf{v}$ is of arity l. Let $\mathcal{B}$ be a finite structure with $\forall_b \exists_{b'}$ as* *a she. For all* $c \in B \setminus \{b, b'\}$*, and for all* $\mathbf{v} := (v_1, \ldots, v_l) \in \{b, b'\}^l$*,*

$$\mathcal{B} \models \varphi(b, \mathbf{v}) \implies \mathcal{B} \models \varphi(c, \mathbf{v}) \implies \mathcal{B} \models \varphi(b', \mathbf{v}).$$

Remark 1. We will only use the restricted version of Lemma 4 that appears as Corollary 1, in which $\mathbf{u}$ is a single variable u. We note that it was necessary for the Lemma's proof that $\mathbf{u}$ might be of greater arity.

We are now ready to state how the presence of $\forall$-, $\exists$- or $\forall\exists$-hyper-operations as shes of $\mathcal{B}$ can diminish the complexity of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$. In each case we proceed by quantifier elimination.

Theorem 3. *If* $\mathcal{B}$ *has a* $\forall$*-operation as a she then* $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$ *is in* NP*. If* $\mathcal{B}$ *has an* $\exists$*-operation as a she then* $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$ *is in* co-NP*. If* $\mathcal{B}$ *has a* $\forall\exists$*-operation as a she then* $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$ *is in* L*.*

Proof. Let φ be a sentence of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}$, and let $\varphi_{[\forall/b]}$ (respectively, $\varphi_{[\exists/b]}$ and $\varphi_{[\forall/b,\exists/b']}$) be φ with all universal variables substituted by b (respectively, existential variables substituted by b and universal variables substituted by b and existential variables substituted by b').

If $\mathcal{B}$ has a she $\forall_b$, then consider a sentence $\varphi \in \{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}$, w.l.o.g. in prenex form. It follows by repeated application of Lemma 2 on φ – either from the outermost quantifier in, or from the innermost quantifier out – that $\mathcal{B} \models \varphi$ iff $\mathcal{B} \models \varphi_{[\forall/b]}$. Similarly, if $\mathcal{B}$ has a she $\exists_{b'}$, then it follows by repeated application of Lemma 3 that $\mathcal{B} \models \varphi$ iff $\mathcal{B} \models \varphi_{[\exists/b']}$.

If $\mathcal{B}$ has a she $\forall_b\exists_{b'}$, then, again, assume the sentence $\varphi \in \{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}$ to be in prenex form. It follows by repeated application of Corollary 1 – from the outermost quantifier in – that $\mathcal{B} \models \varphi$ iff $\mathcal{B} \models \varphi_{[\forall/b,\exists/b']}$. Note that, in this case, one can not move from the innermost quantifier out because this may involve the possibility of free variables taking values from outside the set $\{b, b'\}$. The result now follows since evaluating $\varphi_{[\forall/b,\exists/b']}$ on $\mathcal{B}$ is equivalent to a boolean sentence value problem, known to be in L [**?**]. ☐

Returning to the examples of Figure 3.1, we see that $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{G}_1)$ is in co-NP, $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{G}_2)$ is in NP and both $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{G}_3)$ and $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{G}_4)$ are in L.

3.2 Down-she-monoids of high complexity

Lemma 5. *Let* $\mathcal{B}$*, with* $|B| \geq 2$*, be a structure s.t.* $\mathsf{shE}(\mathcal{B})$ *is a permutation subgroup. Then* $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$ *is* Pspace*-complete.*

Proof. Let $\mathcal{B}_{NAE}$ be the structure on B with a single ternary relation $R_{NAE} := B^3 \setminus \{(b, b, b) : b \in B\}$. $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B}_{NAE})$ is a generalisation of the problem

Figure 2. Further sample digraphs.

$\mathrm{QCSP}(\mathcal{B}_{NAE})$, well-known to be Pspace-complete (in the case $|B| = 2$, this is *quantified not-all-equal* 3*-satisfiability*, see, e.g., [**?**]). $\mathsf{shE}(\mathcal{B}_{NAE})$ is the symmetric group $S_{|B|}$. The statement of the theorem now follows from Theorem 2, since $\mathsf{shE}(\mathcal{B}) \subseteq \mathsf{shE}(\mathcal{B}_{NAE})$. ☐

Corollary 2. *For all* $\mathcal{B}$ *s.t.* $|B| \geq 2$*,* $\{\exists, \forall, \wedge, \vee, =\}$-$\mathsf{FO}(\mathcal{B})$ *is* Pspace*-complete.*

Proof. $\{\exists, \forall, \wedge, \vee, =\}$-$\mathsf{FO}(\mathcal{B})$ may be rephrased as the problem $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B}')$, where $\mathcal{B}'$ is the structure $\mathcal{B}$ expanded with the graph of equality. Owing to the presence of the graph of equality, $\mathsf{shE}(\mathcal{B}')$ must be a permutation subgroup, and the result follows from the previous lemma. ☐

The following is a generalisation of the previous lemma.

Lemma 6. *Let* $\mathcal{B}$ *be a structure whose universe admits the partition* $B_1, \ldots, B_l$ *(*$l \geq 2$*). If all shes of* $\mathcal{B}$ *are sub-hyper-operations of some* f *of the form* $f(x) := B_i$ *iff* $x \in B_{\pi(i)}$*, for* π *a permutation on the set* $\{1, \ldots, l\}$*, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$ *is* Pspace*-complete.*

Proof. Let $\mathcal{K}_{|B_1|,\ldots,|B_l|}$ be the complete l-partitite graph with partitions of size $|B_1|, \ldots, |B_l|$. It may easily be verified that the shes of $\mathcal{K}_{|B_1|,\ldots,|B_l|}$ are of the form of the lemma. Furthermore, $\mathcal{K}_{|B_1|,\ldots,|B_l|}$ agrees with the antireflexive l-clique $\mathcal{K}_l$ on all sentences of equality-free FO logic (for more detail on why this is, see, e.g., the Homomorphism Theorem of [**?**]), and certainly $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}$. Pspace-hardness of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{K}_l)$ follows from the previous lemma, and so Pspace-hardness of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{K}_{|B_1|,\ldots,|B_l|})$ follows a fortiori. Finally, Pspace-hardness of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{B})$ now follows from Theorem 2, since $\mathsf{shE}(\mathcal{B}) \subseteq \mathsf{shE}(\mathcal{K}_{|B_1|,\ldots,|B_l|})$. ☐

In Figure 3.2, four more digraphs $\mathcal{G}_5$–$\mathcal{G}_8$ are drawn. It may easily be verified that $\mathsf{shE}(\mathcal{G}_5)$–$\mathsf{shE}(\mathcal{G}_8)$ areas follows.

$$\begin{array}{cccc} \mathsf{shE}(\mathcal{G}_5) & \mathsf{shE}(\mathcal{G}_6) & \mathsf{shE}(\mathcal{G}_7) & \mathsf{shE}(\mathcal{G}_8) \\ \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array} \right\rangle & \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array} \right\rangle & \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \right\rangle & \left\langle \begin{array}{c|c} 0 & 02 \\ \hline 1 & 1 \\ \hline 2 & 02 \end{array} \right\rangle \end{array}$$

It follows from Lemmas 5 and 6 that each of $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{G}_5)$, $\ldots$, $\{\exists, \forall, \wedge, \vee\}$-$\mathsf{FO}(\mathcal{G}_8)$ is Pspace-complete.

3.3 The boolean case

We consider the case $|B| = 2$, with the normalised domain $B := \{0, 1\}$. It may easily be verified that there are five down-she-monoids in this case, depicted as a lattice in Figure 3.3. The two elements of this lattice that represent the two subgroups of S_2 are drawn in the middle and bottom.

$$\begin{tikzcd}[ampersand replacement=\&]
 \& \underset{\mathsf{L}}{\left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 01 \end{array} \right\rangle} \& \\
\underset{\mathsf{L}}{\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 01 \end{array} \right\rangle} \arrow[ur] \& \underset{\mathsf{Pspace}-c}{\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \right\rangle} \arrow[u] \& \underset{\mathsf{L}}{\left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \end{array} \right\rangle} \arrow[ul] \\
 \& \underset{\mathsf{Pspace}-c}{\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \end{array} \right\rangle} \arrow[ul] \arrow[u] \arrow[ur] \&
\end{tikzcd}$$

Figure 3. The boolean lattice of down-she-monoids with their associated complexity.

Theorem 4 (Dichotomy)**.** *Let $\mathcal{B}$ be a boolean structure.*

I. If either $\forall_0\exists_1$ or $\forall_1\exists_0$ (i.e., $\begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \end{array}$ or $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 01 \end{array}$) is a she of $\mathcal{B}$, then $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{B})$ is in L.

II. Otherwise, $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{B})$ is Pspace-complete.

Proof. $\mathsf{shE}(\mathcal{B})$ must be one of the five down-she-monoids depicted in Figure 3.3. If $\mathsf{shE}(\mathcal{B})$ contains one of $\forall_0\exists_1$ or $\forall_1\exists_0$, then L membership follows from Theorem 3. Otherwise $\mathsf{shE}(\mathcal{B})$ is either $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \end{array} \rangle$ or $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \rangle$; in both cases the hardness result follows from Lemma 5. ☐

Remark 2. In the boolean case, $\langle \forall_1\exists_0 \rangle = \langle \{\forall_1, \exists_0\} \rangle$ and $\langle \forall_0\exists_1 \rangle = \langle \{\forall_0, \exists_1\} \rangle$.

3.4 The three-element case

We consider the case $|B| = 3$, with the normalised domain $B := \{0, 1, 2\}$. In fact, we have the necessary machinery to obtain a full classification theorem. We refer to the six shes

$$\underset{\begin{array}{c|c} 0 & 012 \\ \hline 1 & 1 \\ \hline 2 & 1 \end{array}}{\forall_0\exists_1} \quad \underset{\begin{array}{c|c} 0 & 0 \\ \hline 1 & 012 \\ \hline 2 & 0 \end{array}}{\forall_1\exists_0} \quad \underset{\begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 2 \end{array}}{\forall_0\exists_2} \quad \underset{\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \\ \hline 2 & 012 \end{array}}{\forall_2\exists_0} \quad \underset{\begin{array}{c|c} 0 & 2 \\ \hline 1 & 012 \\ \hline 2 & 2 \end{array}}{\forall_1\exists_2} \quad \underset{\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \\ \hline 2 & 012 \end{array}}{\forall_2\exists_1}$$

as L-*shes*. In Figure 3.4, the lattice of down-she-monoids is partially drawn (all the remaining down-she-monoids contain an L-she). The classification theorem depends on the fact that the lattice is fully drawn on down-she-monoids that do not contain an L-she: this is proved in the Appendix. We remind the reader of the following six shes which also play a role in our classification.

$$\underset{\begin{array}{c|c} 0 & 012 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}}{\forall_0} \quad \underset{\begin{array}{c|c} 0 & 0 \\ \hline 1 & 012 \\ \hline 2 & 2 \end{array}}{\forall_1} \quad \underset{\begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 012 \end{array}}{\forall_2} \quad \underset{\begin{array}{c|c} 0 & 0 \\ \hline 1 & 01 \\ \hline 2 & 02 \end{array}}{\exists_0} \quad \underset{\begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}}{\exists_1} \quad \underset{\begin{array}{c|c} 0 & 02 \\ \hline 1 & 12 \\ \hline 2 & 2 \end{array}}{\exists_2}$$

Theorem 5 (Tetrachotomy)**.** *Let $\mathcal{B}$ be a three-element structure.*

I. If $\mathsf{shE}(\mathcal{B})$ contains any of the L-shes, then $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{B})$ is in L.

II. If $\mathsf{shE}(\mathcal{B})$ contains none of the L-shes, but contains one of $\forall_0$, $\forall_1$ or $\forall_2$, then $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{B})$ is NP-complete.

III. If $\mathsf{shE}(\mathcal{B})$ contains none of the L-shes, but contains one of $\exists_0$, $\exists_1$ or $\exists_2$, then $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{B})$ is $\mathsf{co\text{-}NP}$-complete.

IV. Otherwise, $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{B})$ is Pspace-complete.

Proof. The L case, Case I, follows from Theorem 3, as does membership of NP and $\mathsf{co\text{-}NP}$, for Cases II and III, respectively.

For NP-hardness in Case II, consider the disjoint union $\mathcal{K}_2 \uplus \mathcal{K}_1$, whose shes constitute one of the down-she-monoids $\langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \rangle$, $\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 012 \\ \hline 2 & 0 \end{array} \rangle$ or $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 012 \end{array} \rangle$, depending on the vertex labelling. $\{\exists,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{K}_2)$ is NP-complete (by reduction from 3-*not-all-equal satisfiablity*, set $R_{NAE}(u, v, w) := E(u, v) \vee E(v, w)$), and $\mathcal{K}_2 \uplus \mathcal{K}_1$ agrees with $\mathcal{K}_2$ on all sentences of $\{\exists,\wedge,\vee\}$-FO (see [**?**]). It follows that $\{\exists,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{K}_2 \uplus K_1)$ is NP-complete and that $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{K}_2 \uplus K_1)$ is NP-hard. The result for NP-hardness now follows from Lemma 5.

For a graph $\mathcal{G}$, define its complement $\overline{\mathcal{G}}$ over the same vertex set to have the complementary edge set (i.e. $E(x, y) \in \overline{\mathcal{G}}$ iff $E(x, y) \notin \mathcal{G}$). It is a simple application of de Morgan duality that $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{G})$ is in NP (resp., is NP-complete) iff $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\overline{\mathcal{G}})$ is in $\mathsf{co\text{-}NP}$ (resp., is $\mathsf{co\text{-}NP}$-complete) – see [**?**]. A similar argument to that in the previous paragraph, with the complement graph $\overline{\mathcal{K}_2 \uplus \mathcal{K}_1}$ yields the $\mathsf{co\text{-}NP}$-hardness result for Class III.

Finally, since all other down-she-monoids are of the form of either Lemmas 5 or 6, the Pspace-hardness results follow for Class IV. ☐

Casting our mind back to the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 3.1, we can read from the previous theorem that $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{G}_1)$ and $\{\exists,\forall,\wedge,\vee\}$-$\mathsf{FO}(\mathcal{G}_2)$ are $\mathsf{co\text{-}NP}$-complete and NP-complete, respectively.

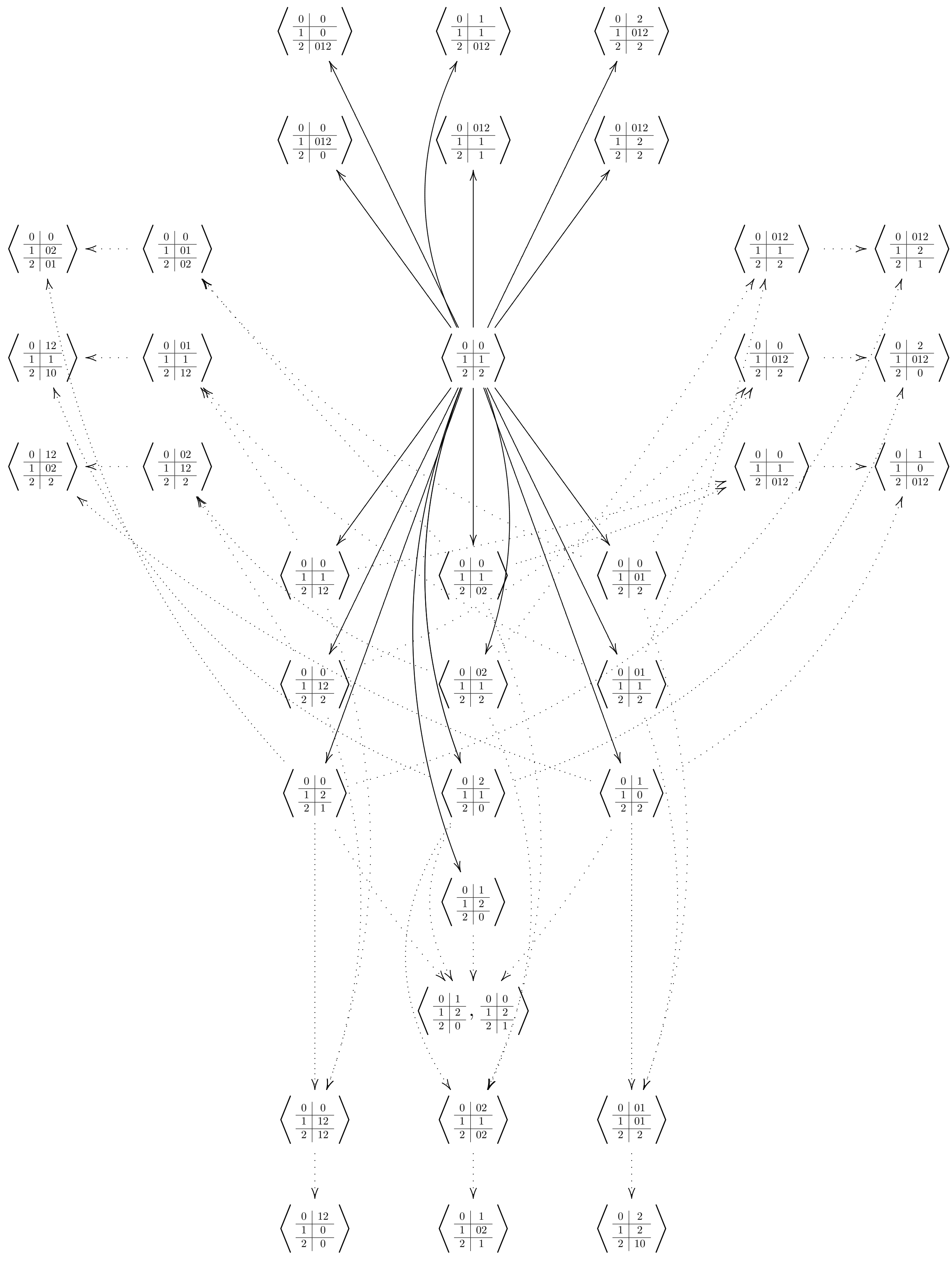

Figure 4. Part of the lattice of down-she-monoids in the three-element case. All the remaining down-she-monoids contain one of the six L-shes.

4 Final remarks

We have introduced the class of problems $\{\exists, \forall, \wedge, \vee\}$-FO($\mathcal{B}$) as well as an algebraic framework in which to study their complexity. We hope that we have adequately demonstrated that this class of problems displays complexity-theoretic richness, while not being too resistant to full classification in simple cases. The algebraic method used in our classification for the three-element case gives simple explanation where there previously was none – if one were to look at the examples of Figures 3.1 and 3.2, there is little obvious in their immediate structure that betrays their position in the classification.

We note that our positive algorithms, for membership of NP, co-NP and – especially – L, are uniform, and are based on simple quantifier elimination. Perhaps it is to be hoped that a full classification for the problems $\{\exists, \forall, \wedge, \vee\}$-FO($\mathcal{B}$) would make use only of versions of quantifier elimination. In any case, we conjecture that the tetrachotomy of Theorem 5 extends to all structures $\mathcal{B}$; though we know we would need more sophistocated classes of shes than those of Section 3.1 to prove this. We note also that, unlike the situation with clones and the CSP, the down-she-monoids associated with a finite domain are always finite. This means that their lattice should be actually computable for low domain sizes like four or five.

5 Appendix

Throughout this section we consider shes only on the set $\{0, 1, 2\}$. As before, we refer to the following six shes as L-*shes*: $\begin{array}{c|c}0&0\\\hline 1&0\\\hline 2&012\end{array}$, $\begin{array}{c|c}0&0\\\hline 1&012\\\hline 2&0\end{array}$, $\begin{array}{c|c}0&012\\\hline 1&1\\\hline 2&1\end{array}$, $\begin{array}{c|c}0&012\\\hline 1&2\\\hline 2&2\end{array}$, $\begin{array}{c|c}0&1\\\hline 1&1\\\hline 2&012\end{array}$, $\begin{array}{c|c}0&2\\\hline 1&012\\\hline 2&2\end{array}$. Recall that the three shes $\begin{array}{c|c}0&012\\\hline 1&0\\\hline 2&0\end{array}$, $\begin{array}{c|c}0&1\\\hline 1&012\\\hline 2&1\end{array}$ and $\begin{array}{c|c}0&2\\\hline 1&2\\\hline 2&012\end{array}$ each generate $\langle \begin{array}{c|c}0&012\\\hline 1&012\\\hline 2&012\end{array} \rangle$, which contains all the L-shes.

We will firstly undertake a systematic study of down-she-monoids generated by a single she. We say that a she f *omits the* L-*shes* if $\langle f \rangle$ does not contain any of the L-shes. It may be verified that there are 265 shes on $\{0, 1, 2\}$. We will discover that only 48 of them omit the L-shes. Furthermore, of the 48, there are actually only 29 shes f s.t. $\langle f \rangle$ are inequivalent. Later, we shall see that these down-she monoids are (almost) the only ones that do not contain an L-she.

We will progress through the shes by *type*. A type is a three-element multiset over the base set $\{1, 2, 3\}$; for example a she f is of type $\{1; 2; 2\}$[7] if either $|f(0)| = 2, |f(1)| = 1, |f(2)| = 1$ or $|f(0)| = 1, |f(1)| = 2, |f(2)| = 1$ or $|f(0)| = 1, |f(1)| = 1, |f(2)| = 2$.

Shes of type $\{1; 1; 1\}$. There a 6 shes of this type, including the identity. These correspond to elements of the symmetric group S_3. Each of these omits the L-shes.

Shes of type $\{2; 1; 1\}$**.** There are 45 shes of this type. We will establish that 27 of them omit the L-shes. We will draw only the case where $|f(0)| = 2$; the cases $|f(1)| = 2$ and $|f(2)| = 2$ follow by symmetry. In the following, the first row depicts the she f under consideration; the second row depicts f^2. In two cases it is necessary to derive f^3 in the third row to demonstrate containment of an L-she in $\langle f \rangle$. The complexity of the respective down-she-monoids is written in the fourth row as a guide; L is written exactly where the down-she-monoid contains one of the L-shes.

$$\begin{array}{cc||l|l|l|l|l}
 & 0 & 01 & 01 & 01 & 01 & 01\\
f & 1 & 0 & 1 & 2 & 2 & 2\\
 & 2 & 2 & 2 & 0 & 1 & 2\\
 & & & & & & \\
 & 0 & 01 & 01 & 012 & 012 & 012\\
f^2 & 1 & 01 & 1 & 0 & 1 & 2\\
 & 2 & 2 & 2 & 01 & 2 & 2\\
 & & & & & & \\
 & & Psp & Psp & L & NP & L
\end{array}$$

$$\begin{array}{cc||l|l|l|l|l}
 & 0 & 02 & 02 & 02 & 02 & 02\\
f & 1 & 0 & 1 & 1 & 1 & 2\\
 & 2 & 1 & 0 & 1 & 2 & 1\\
 & & & & & & \\
 & 0 & 012 & 02 & 012 & 02 & 012\\
f^2 & 1 & 02 & 1 & 1 & 1 & 1\\
 & 2 & 0 & 02 & 1 & 2 & 2\\
 & & & & & & \\
 & & L & Psp & L & Psp & NP
\end{array}$$

$$\begin{array}{cc||l|l|l|l|l}
 & 0 & 12 & 12 & 12 & 12 & 12\\
f & 1 & 0 & 0 & 0 & 1 & 2\\
 & 2 & 0 & 1 & 2 & 0 & 0\\
 & & & & & & \\
 & 0 & 0 & 01 & 02 & 01 & 02\\
f^2 & 1 & 12 & 12 & 12 & 1 & 0\\
 & 2 & 12 & 0 & 2 & 12 & 12\\
 & & & & & & \\
 & 0 & & 012 & & & 012\\
f^3 & 1 & & 01 & & & 12\\
 & 2 & & 12 & & & 02\\
 & & & & & & \\
 & & Psp & L & coNP & coNP & L
\end{array}$$

It is readily seen (and was in any case obvious) that the cases $f(0) = \{0, 1\}$ and $f(0) := \{0, 2\}$ are actually symmetric.

Shes of type $\{1; 2; 2\}$. There are 63 shes of this type. We will establish that only 9 of them omit the L-shes. We will draw only the case where $|f(0)| = 1$; the cases $|f(1)| = 1$

[7] We separate elements by ';' to indicate a multiset.

and $|f(2)| = 1$ follow by symmetry.

f	0	1	1	1	1	1	1	1
	1	01	01	02	02	02	12	12
	2	02	12	01	02	12	01	02
f^2	0	01	01	02	02	02	12	12
	1	01	01	01	012	12	012	012
	2	012	012	012	012	012	12	012
		L	L	L	L	L	L	L

f	0	2	2	2	2	2	2	2
	1	01	01	01	02	02	12	12
	2	01	02	12	01	12	01	02
f^2	0	01	02	12	01	12	01	02
	1	012	012	012	012	12	012	012
	2	012	02	012	02	012	12	02
		L	L	L	L	L	L	L

f	0	0	0	0	0	0	0	0
	1	01	01	02	02	12	12	12
	2	02	12	01	12	01	02	12
f^2	0	0	0	0	0	0	0	0
	1	01	01	01	012	012	012	12
	2	02	012	02	012	012	02	12
		$coNP$	L	$coNP$	L	L	L	Psp

Again, the cases $f(0) = \{1\}$ and $f(0) = \{2\}$ are actually symmetric.

Shes of type $\{3; 1; 1\}$. There are 27 shes of this type, but only 6 omit the L-shes. We will draw only the case where $|f(0)| = 3$; the cases $|f(1)| = 3$ and $|f(2)| = 3$ follow by symmetry.

f	0	012	012	012	012	012	012	012	012	012
	1	0	0	0	1	1	1	2	2	2
	2	0	1	2	0	1	2	0	1	2
f^2	0		012	012	012		012	012	012	
	1		012	012	1		1	0	1	
	2		0	2	012		2	012	2	
		L	L	L	L	L	NP	L	NP	L

Shes of type $\{2; 2; 2\}$. None of these omits the L-shes. We can demonstrate this by considering extensions of the few of type $\{1; 2; 2\}$ that themselves omit the L-shes. In the following, we consider extensions of the three $\{1; 2; 2\}$ types in which $|f(0)| = 1$ that omit the L-shes. The remaining cases follow by symmetry. The first row depicts the three $\{1; 2; 2\}$ types, with $|f(0)| = 1$, that omit the L-shes; the second row their extensions to type $\{2; 2; 2\}$ (each has two). The third row gives the extension squared, in order to show containment of an L-she.

	0	0		0		0	
	1	01		02		12	
	2	02		01		12	
f	0	01	02	01	02	01	02
	1	01	01	02	02	12	12
	2	02	02	01	01	12	12
f^2	0	01	02	012	012	012	012
	1	01	012	01	012	12	12
	2	012	02	012	02	12	12

Shes of type $\{3; 2; 1\}$. None of these omits the L-shes. We can demonstrate this by considering extensions of the few of type $\{3; 1; 1\}$ that themselves omit the L-shes. In the following, we consider extensions of the two $\{3; 1; 1\}$ types in which $|f(0)| = 3$ that omit the L-shes. The remaining cases follow by symmetry. The first row depicts the two $\{3; 1; 1\}$ types that omit the L-shes; the second row their extensions to type $\{3; 2; 1\}$ (each has four). The third row gives the extension squared, where this is necessary to show containment of an L-she.

	0	012				012			
	1	1				2			
	2	2				1			
f	0	012	012	012	012	012	012	012	012
	1	01	12	1	1	02	12	2	2
	2	2	2	02	12	1	1	01	12
f^2	0	012		012		012		012	
	1	012		1		012		01	
	2	2		012		02		012	

Shes of any other type. None of these omits the L-shes as these must contain one of $\{2; 2; 2\}$ or $\{3; 2; 1\}$ as a subtype.

5.1 Duplications.

It is a trivial observation that $\langle \begin{smallmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{smallmatrix} \rangle = \langle \begin{smallmatrix} 0 & 2 \\ 1 & 0 \\ 2 & 1 \end{smallmatrix} \rangle$. It may therefore be verified that there are 5 inequivalent down-she-monoids generated by singletons of type $\{1; 1; 1\}$. Furthermore, various of the type $\{2; 1; 1\}$ shes are such that they generate the same down-she-monoids as some of type $\{1; 2; 2\}$ or $\{3; 1; 1\}$. We will state all of the relationships that concern us; their proofs are left as a simple exercise.

$$\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 1 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 12 \end{array} \right\rangle$$
$$\left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 01 \\ \hline 2 & 2 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 0 \\ \hline 2 & 2 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 01 \\ \hline 2 & 2 \end{array} \right\rangle$$
$$\left\langle \begin{array}{c|c} 0 & 02 \\ \hline 1 & 1 \\ \hline 2 & 02 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 02 \\ \hline 1 & 1 \\ \hline 2 & 0 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 1 \\ \hline 2 & 02 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 02 \\ \hline 2 & 01 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 02 \\ \hline 2 & 1 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 01 \end{array} \right\rangle$$
$$\left\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 02 \\ \hline 2 & 2 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 02 \\ \hline 2 & 2 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 2 \end{array} \right\rangle$$
$$\left\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 1 \\ \hline 2 & 01 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 1 \\ \hline 2 & 0 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 1 \\ \hline 2 & 01 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 02 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \right\rangle$$
$$\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 012 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 02 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 12 \end{array} \right\rangle$$
$$\left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 012 \\ \hline 2 & 0 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 01 \\ \hline 2 & 0 \end{array} \right\rangle = \left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 12 \\ \hline 2 & 0 \end{array} \right\rangle$$

It follows that we may discard 18 of the 27 of type $\{2;1;1\}$. We are left with only 29 shes whose interactions we need to consider in the analysis of down-she-monoids that do not contain a L-shes: 5 of type $\{1;1;1\}$ (including the identity), 9 of type $\{2;1;1\}$, 9 of type $\{1;2;2\}$ and 6 of type $\{3;1;1\}$. We depict the 29 cases (28 plus the identity) in the following manner for future reference. We classify the 28 into five classes A-E, each of six, except D of five.

5.2 Combinations of the 28.

We now systematically consider the ways in which the 28 shes combine with one another.

5.2.1 Combinations involving Class D.

Combinations involving a cyclic permutation. We consider shes of the form $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \\ \hline 2 & 0 \end{array}, f\rangle$. If f is in $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \\ \hline 2 & 0 \end{array}\rangle$, then we generate $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \\ \hline 2 & 0 \end{array}\rangle$. If f is any of the transpositions $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}$, $\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 2 \end{array}$ or $\begin{array}{c|c} 0 & 2 \\ \hline 1 & 1 \\ \hline 2 & 0 \end{array}$, then we generate the down-she-monoid corresponding to S_3: $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \\ \hline 2 & 0 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}\rangle$. We will discover that this is the only down-she-monoid, other than the 29 generated by singletons, that does not contain an L-she. As a first step to this result, we now prove that, if f is any other she (than type $\{1;1;1\}$), then $\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \\ \hline 2 & 0 \end{array}, f\rangle$ contains an L-she. It is sufficient to prove this for f of type $\{2;1;1\}$ where $\langle f\rangle$ does not already contain an L-she. We consider only the case where $|f(0)| = 2$; the others being symmetric. Since the cases $f(0) = \{0,1\}$ and $f(0) := \{0,2\}$ are symmetric, we consider only the former. In the following table, from the third row, each entry is the composition of the she two above it on the she immediately above it, with the derivation ceasing when containment of an L-she is obvious.[8]

e	0	1	1	1	1	1	1
	1	2	2	2	2	2	2
	2	0	0	0	0	0	0
f	0	01	01	01	12	12	12
	1	0	1	2	0	0	1
	2	2	2	1	0	2	0
$g := e \circ f$	0	12	12	12	02	02	02
	1	1	2	0	1	1	2
	2	0	0	2	1	0	1
$h := f \circ g$	0	02	12	12	012	12	012
	1	0	2	01	0	0	0
	2	01	01	1	0	12	1
$i := g \circ h$	0	012	02	02		01	012
	1	12	0	012		02	02
	2	12	12	0		01	2
$j := h \circ i$	0		012			012	
	1		12			12	
	2		012			012	

Combinations involving a transposition. We consider shes of the form $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}, f\rangle$, where f is one of our 28 (of course, the similar cases for $\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 2 \end{array}$ and $\begin{array}{c|c} 0 & 2 \\ \hline 1 & 1 \\ \hline 2 & 0 \end{array}$ are symmetric). When $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}$ is combined with anything in $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}\rangle$, clearly $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}\rangle$ results. When $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}$ is combined with any of $\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 2 \end{array}$, $\begin{array}{c|c} 0 & 2 \\ \hline 1 & 1 \\ \hline 2 & 0 \end{array}$ or $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}$ then the down-she-monoid equivalent to S_3 results: $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}\rangle$.

We consider the remaining 24 cases in groups of six. For the Class C, we specify the resultant down-she monoid (when in combination with $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}$).

0	0	0	0	0	02	01
1	1	1	01	12	1	1
2	12	02	2	2	2	2
	$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}\rangle$	$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 02 \\ \hline 2 & 01 \end{array}\rangle$	$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 02 \\ \hline 2 & 01 \end{array}\rangle$	$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}\rangle$	$\langle \begin{array}{c\|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}\rangle$	$\langle \begin{array}{c\|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}\rangle$

For Classes A,B and E, we specify either the resultant down-she-monoid or we give a derivation of an L-she: from

[8] Note that this form of derivation will recur hereonin. Remember that f on g involves evaluating first g then f.

$$
\begin{array}{cccccccc}
\left\langle\begin{array}{c|c}0&0\\\hline 1&02\\\hline 2&01\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&0\\\hline 1&01\\\hline 2&02\end{array}\right\rangle & \Leftarrow & \text{Class A} & & \left\langle\begin{array}{c|c}0&012\\\hline 1&1\\\hline 2&2\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&012\\\hline 1&2\\\hline 2&1\end{array}\right\rangle \\
\left\langle\begin{array}{c|c}0&12\\\hline 1&1\\\hline 2&10\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&01\\\hline 1&1\\\hline 2&12\end{array}\right\rangle & \swarrow & \left\langle\begin{array}{c|c}0&0\\\hline 1&1\\\hline 2&2\end{array}\right\rangle & \nearrow & \left\langle\begin{array}{c|c}0&0\\\hline 1&012\\\hline 2&2\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&2\\\hline 1&012\\\hline 2&0\end{array}\right\rangle \\
\left\langle\begin{array}{c|c}0&12\\\hline 1&02\\\hline 2&2\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&02\\\hline 1&12\\\hline 2&2\end{array}\right\rangle & & \text{Class B} & \Rightarrow & \left\langle\begin{array}{c|c}0&0\\\hline 1&1\\\hline 2&012\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&1\\\hline 1&0\\\hline 2&012\end{array}\right\rangle
\end{array}
$$

$$
\begin{array}{cccccccc}
\text{Class C} & \Rightarrow & \left\langle\begin{array}{c|c}0&0\\\hline 1&1\\\hline 2&12\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&0\\\hline 1&1\\\hline 2&02\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&0\\\hline 1&01\\\hline 2&2\end{array}\right\rangle & & \\
& \Rightarrow & \left\langle\begin{array}{c|c}0&0\\\hline 1&12\\\hline 2&2\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&02\\\hline 1&1\\\hline 2&2\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&01\\\hline 1&1\\\hline 2&2\end{array}\right\rangle & & \\
& & \left\langle\begin{array}{c|c}0&0\\\hline 1&2\\\hline 2&1\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&2\\\hline 1&1\\\hline 2&0\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&1\\\hline 1&0\\\hline 2&2\end{array}\right\rangle & \Leftarrow & \text{Class D} \\
& & & \left\langle\begin{array}{c|c}0&1\\\hline 1&2\\\hline 2&0\end{array}\right\rangle & & \Leftarrow & \\
\text{Class E} & \Rightarrow & \left\langle\begin{array}{c|c}0&0\\\hline 1&12\\\hline 2&12\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&02\\\hline 1&1\\\hline 2&02\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&01\\\hline 1&01\\\hline 2&2\end{array}\right\rangle & & \\
& \Rightarrow & \left\langle\begin{array}{c|c}0&12\\\hline 1&0\\\hline 2&0\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&1\\\hline 1&02\\\hline 2&1\end{array}\right\rangle & \left\langle\begin{array}{c|c}0&2\\\hline 1&2\\\hline 2&10\end{array}\right\rangle & &
\end{array}
$$

the third row each entry is the composition of the she two above it on the she immediately above it.

e	0	0	0	0	0	0	0
	1	2	2	2	2	2	2
	2	1	1	1	1	1	1
f	0	0	01	02	0	12	12
	1	01	1	12	02	1	02
	2	02	12	2	01	01	2
$g := e \circ f$	0		02	01		12	12
	1		2	12		2	01
	2		12	1		02	1
$h := f \circ g$	0		012	012		01	02
	1		12	12		01	012
	2		12	12		012	02
		$\left\langle\begin{array}{c\|c}0&0\\\hline 1&02\\\hline 2&01\end{array}\right\rangle$			$\left\langle\begin{array}{c\|c}0&0\\\hline 1&02\\\hline 2&01\end{array}\right\rangle$		

e	0	0	0	0	0	0	0
	1	2	2	2	2	2	2
	2	1	1	1	1	1	1
f	0	012	0	0	012	2	1
	1	1	012	1	2	012	0
	2	2	2	012	1	0	012
$g := e \circ f$	0		0	0		1	2
	1		012	2		012	0
	2		1	012		0	012
$h := f \circ g$	0		0	0		012	012
	1		012	012		012	1
	2		012	012		2	012
		$\left\langle\begin{array}{c\|c}0&012\\\hline 1&2\\\hline 2&1\end{array}\right\rangle$			$\left\langle\begin{array}{c\|c}0&012\\\hline 1&2\\\hline 2&1\end{array}\right\rangle$		

e	0	0	0	0	0	0	0
	1	2	2	2	2	2	2
	2	1	1	1	1	1	1
f	0	0	02	01	12	1	2
	1	12	1	01	0	02	2
	2	12	02	2	0	1	10
$g := e \circ f$	0		01	02		2	1
	1		2	02		01	1
	2		01	1		2	02
$h := f \circ g$	0		012	012		1	2
	1		02	012		012	2
	2		012	01		1	012
		$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \rangle$			$\langle \begin{array}{c\|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array} \rangle$		

5.2.2 Combinations involving Class E.

Without loss of generality, we consider only $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}$ and $\begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}$ in combination with shes of Classes A,B,C and E (the other cases being symmetric or covered in the previous section). Since $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \rangle \subseteq \langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \rangle \subseteq \langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array} \rangle$, we only consider, for $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}$ (respectively, $\begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}$) shes f s.t. $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}, f \rangle$ (respectively, $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}, f \rangle$) does not contain an L-she. We begin with combinations of $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}$ on Class C. In each case, we specify either the resultant down-she-monoid or we give a derivation of an L-she: from the third row each entry is the composition of the she two above it on the she immediately above it.

e	0	0	0	0	0	0	0
	1	12	12	12	12	12	12
	2	12	12	12	12	12	12
f	0	0	0	0	0	02	01
	1	1	1	01	12	1	1
	2	12	02	2	2	2	2
$g := e \circ f$	0		0	0		012	012
	1		12	012		12	12
	2		012	12		12	12
$h := f \circ g$	0		0	0			
	1		012	012			
	2		012	012			
		$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \rangle$			$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \rangle$		

We are left with only two of the above to consider in combination with $\begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}$. We state the equivalence

$$\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \rangle = \langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 2 \end{array} \rangle = \langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array} \rangle.$$

Now we consider the combination of $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array}$ with the six shes f of Classes A, B and E that are s.t. $\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}, f \rangle$ does not contain an L-she. In each case, we specify either the resultant down-she-monoid or we give a derivation of an L-she: from the third row each entry is the composition of the she two above it on the she immediately above it.

e	0	0	0	0	0	0	0
	1	12	12	12	12	12	12
	2	12	12	12	12	12	12
f	0	0	0	012	012	0	12
	1	01	02	1	2	12	0
	2	02	01	2	1	12	0
$g := e \circ f$	0	0	0	012	012		
	1	012	012	12	12		
	2	012	012	12	12		
						$\langle \begin{array}{c\|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \rangle$	$\langle \begin{array}{c\|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array} \rangle$

We are then left with two of the above to consider in combination with $\begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}$. We state the trivialities

$$\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 12 \\ \hline 2 & 12 \end{array} \rangle = \langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array}, \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array} \rangle = \langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \end{array} \rangle.$$

5.2.3 Combinations involving Class C.

Without loss of generality, we consider only $\begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}$ in combination with shes of Classes A-C (the other cases being symmetric or covered in the previous sections). We will consider the eighteen shes in groups of six. In each case, we specify either the resultant down-she-monoid or we give a derivation of an L-she: from the third row each entry is the composition of the she two above it on the she immediately above it.

	0	0	0	0	0	0	0
e	1	1	1	1	1	1	1
	2	12	12	12	12	12	12
	0	0	01	02	0	12	12
f	1	01	1	12	02	1	02
	2	02	12	2	01	10	2
	0	0		012	0		12
$g := e \circ f$	1	01		12	012		012
	2	012		12	01		12
			$\langle \begin{smallmatrix} 0 & 01 \\ 1 & 1 \\ 2 & 12 \end{smallmatrix} \rangle$			$\langle \begin{smallmatrix} 0 & 12 \\ 1 & 1 \\ 2 & 01 \end{smallmatrix} \rangle$	

	0	0	0	0	0	0	0
e	1	1	1	1	1	1	1
	2	12	12	12	12	12	12
	0	012	0	0	012	2	1
f	1	1	012	1	2	012	0
	2	2	2	012	1	0	012
	0	012	0		012	12	
$g := e \circ f$	1	1	012		12	012	
	2	12	12		1	0	
	0		0			012	
$h := f \circ g$	1		012			012	
	2		012			2	
				$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 012 \end{smallmatrix} \rangle$			$\langle \begin{smallmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 012 \end{smallmatrix} \rangle$

	0	0	0	0	0	0	0
e	1	1	1	1	1	1	1
	2	12	12	12	12	12	12
	0	0	0	0	0	02	01
f	1	1	1	01	12	1	1
	2	12	02	2	2	2	12
	0			0		012	
$g :=$	1			01		1	
$e \circ f$	2			12		12	
	0			0			
$h :=$	1			01			
$f \circ g$	2			012			
		$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 12 \end{smallmatrix} \rangle$	$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 012 \end{smallmatrix} \rangle$		$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 12 \\ 2 & 12 \end{smallmatrix} \rangle$		$\langle \begin{smallmatrix} 0 & 01 \\ 1 & 1 \\ 2 & 12 \end{smallmatrix} \rangle$

5.2.4 Intercombinations of Classes A and B.

Let f and g be Class A and Class B shes, respectively, s.t. b' is the element in each of the image sets $f(0), f(1), f(2)$ and b is the element s.t. $g(b) = \{0, 1, 2\}$ – i.e. $f = \exists_{b'}$ and $g = \forall_b$. It is an elementary matter to verify that $\langle f, g\rangle$ necessarily contains an L-she (it contains the she h that maps b to $\{0, 1, 2\}$ and everything else to b'). We are left only to consider the combinations of Class A (respectively, Class B) on itself.

For Class A, and w.l.o.g., we consider only $\begin{smallmatrix} 0 & 0 \\ 1 & 01 \\ 2 & 02 \end{smallmatrix}$ and $\begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix}$ with the others. Since $\langle \begin{smallmatrix} 0 & 0 \\ 1 & 01 \\ 2 & 02 \end{smallmatrix} \rangle \subseteq \langle \begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix} \rangle$, we will consider the latter case only with those that do not generate an L-she with the former case. In each case, we specify either the resultant down-she-monoid or we give a derivation of an L-she: from the third row each entry is the composition of the she two above it on the she immediately above it.

	0	0	0	0	0	0	0
e	1	01	01	01	01	01	01
	2	02	02	02	02	02	02
	0	0	01	02	0	12	12
f	1	01	1	12	02	1	02
	2	02	12	2	01	01	2
	0		01	02		012	012
$g := e \circ f$	1		01	012		01	02
	2		012	02		01	02
		$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 01 \\ 2 & 02 \end{smallmatrix} \rangle$			$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix} \rangle$		

We can now give the trivialities

$$\langle \begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 01 \\ 2 & 02 \end{smallmatrix} \rangle = \langle \begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix} \rangle = \langle \begin{smallmatrix} 0 & 0 \\ 1 & 02 \\ 2 & 01 \end{smallmatrix} \rangle.$$

For Class B, we procede in a similar manner.

	0	012	012	012	012	012	012
e	1	1	1	1	1	1	1
	2	2	2	2	2	2	2
	0	012	0	0	012	2	1
f	1	1	012	1	2	012	0
	2	2	2	012	1	0	012
	0		012	012		2	1
$g := e \circ f$	1		012	1		012	012
	2		2	012		012	012
		$\langle \begin{smallmatrix} 0 & 012 \\ 1 & 1 \\ 2 & 2 \end{smallmatrix} \rangle$			$\langle \begin{smallmatrix} 0 & 012 \\ 1 & 2 \\ 2 & 1 \end{smallmatrix} \rangle$		

Finally, the trivialities

$$\langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}, \begin{array}{c|c} 0 & 012 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array} \rangle = \langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array}, \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \rangle = \langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \end{array} \rangle.$$