Second-Order Uniformly Convergent Hybrid Numerical Scheme for Singularly Perturbed Problems of Mixed Parabolic-Elliptic Type

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1. Model Problem
2. Properties of Analytical Solution
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2 Properties of Analytical Solution

3 The Numerical Scheme

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3 The Numerical Scheme
4 Error Analysis

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Model Problem

Denote the domains by \( \Omega = (0, 1) \), \( \Omega^- = (0, \xi) \), \( \Omega^+ = (\xi, 1) \), and \( G = \Omega \times (0, T] \), \( G^- = \Omega^- \times (0, T] \), \( G^+ = \Omega^+ \times (0, T] \).

Consider the singularly perturbed mixed parabolic-elliptic problems posed on the domain \( G^- \cup G^+ \):

\[
\begin{align*}
L_{1,\varepsilon} u(x, t) &\equiv \left( \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + b(x, t)u \right)(x, t) = f(x, t), \quad (x, t) \in G^- , \\
L_{2,\varepsilon} u(x, t) &\equiv \left( -\varepsilon \frac{\partial^2 u}{\partial x^2} - a(x, t) \frac{\partial u}{\partial x} + b(x, t)u \right)(x, t) = f(x, t), \quad (x, t) \in G^+ , \\
u(x, 0) & = s_0(x), \quad x \in \Omega , \\
u(0, t) & = s_1(t), \quad u(1, t) = s_2(t), \quad t \in (0, T],
\end{align*}
\]

(1)
Model Problem

Denote the domains by $\Omega = (0, 1)$, $\Omega^- = (0, \xi)$, $\Omega^+ = (\xi, 1)$, and $G = \Omega \times (0, T]$, $G^- = \Omega^- \times (0, T]$, $G^+ = \Omega^+ \times (0, T]$.

Consider the singularly perturbed mixed parabolic-elliptic problems posed on the domain $G^- \cup G^+$:

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L_{2,\varepsilon}u(x, t) &\equiv \left( -\varepsilon \frac{\partial^2 u}{\partial x^2} - a(x, t) \frac{\partial u}{\partial x} + b(x, t)u \right) (x, t) = f(x, t), \quad (x, t) \in G^+,

u(x, 0) &= s_0(x), \quad x \in \overline{\Omega},

u(0, t) &= s_1(t), \quad u(1, t) = s_2(t), \quad t \in (0, T],
\end{aligned}
$$

subject to the interface conditions

$$
[u] = 0, \quad \left[ \frac{\partial u}{\partial x} \right] = 0, \quad \text{at } x = \xi.
$$

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Model Problem

Denote the domains by $\Omega = (0, 1)$, $\Omega^- = (0, \xi)$, $\Omega^+ = (\xi, 1)$, and $G = \Omega \times (0, T)$, $G^- = \Omega^- \times (0, T)$, $G^+ = \Omega^+ \times (0, T)$.

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\begin{align*}
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L_{2, \varepsilon} u(x, t) &\equiv \left( -\varepsilon \frac{\partial^2 u}{\partial x^2} - a(x, t) \frac{\partial u}{\partial x} + b(x, t) u \right)(x, t) = f(x, t), \quad (x, t) \in G^+, \\
u(x, 0) &= s_0(x), \quad x \in \Omega, \\
u(0, t) &= s_1(t), \quad u(1, t) = s_2(t), \quad t \in (0, T),
\end{align*}
$$

(1)

subject to the interface conditions

$$
[u] = 0, \quad \left[ \frac{\partial u}{\partial x} \right] = 0, \quad \text{at } x = \xi.
$$

(2)

$0 < \varepsilon \ll 1$ is a small parameter.

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Define the jump of $u$, denoted by $[u]$, across the point of discontinuity $x = \xi$ by $[u](\xi, t) = u(\xi+, t) - u(\xi-, t)$. 
Model Problem

- Define the jump of $u$, denoted by $[u]$, across the point of discontinuity $x = \xi$ by $[u](\xi, t) = u(\xi^+, t) - u(\xi^-, t)$.
- Assume that the coefficients satisfy the conditions:

\[
\begin{aligned}
\begin{cases}
  b(x, t) \geq \beta \geq 0 & \text{on } \overline{G}, \\
  \alpha^* > a(x, t) > \alpha > 0, & x > \xi.
\end{cases}
\end{aligned}
\]
Model Problem

- Define the jump of \( u \), denoted by \([u]\), across the point of discontinuity \( x = \xi \) by \([u](\xi, t) = u(\xi^+, t) - u(\xi^-, t)\).
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\begin{align*}
  b(x, t) &\geq \beta \geq 0 \quad \text{on } \overline{G}, \\
  \alpha^* &> a(x, t) > \alpha > 0, \quad x > \xi.
\end{align*}
\]

- Since the equation in (1) is of convection-diffusion type in \( G^+ \) and the convection coefficient is positive in \( \Omega^+ \), an interior layer of width \( O(\varepsilon) \) appears on the left side of the boundary of \( G^+ \).
Define the jump of $u$, denoted by $[u]$, across the point of discontinuity $x = \xi$ by $[u](\xi, t) = u(\xi^+, t) - u(\xi^-, t)$.

Assume that the coefficients satisfy the conditions:

$$\begin{cases} b(x, t) \geq \beta \geq 0 \quad \text{on } \overline{G}, \\ \alpha^* > a(x, t) > \alpha > 0, \quad x > \xi. \end{cases}$$

Since the equation in (1) is of convection-diffusion type in $G^+$ and the convection coefficient is positive in $\Omega^+$, an interior layer of width $O(\varepsilon)$ appears on the left side of the boundary of $G^+$.

Whereas the equation in (1) is of parabolic reaction-diffusion type in $G^-$, which leads to occurrence of parabolic boundary layers of width $O(\sqrt{\varepsilon})$ on both the boundaries of $G^-$. 

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Model Problem

- Define the jump of \( u \), denoted by \([u]\), across the point of discontinuity \( x = \xi \) by 
  \[ [u](\xi, t) = u(\xi^+, t) - u(\xi^-, t). \]

- Assume that the coefficients satisfy the conditions:
  \[
  \begin{cases}
  b(x, t) \geq \beta \geq 0 & \text{on } \overline{G}, \\
  \alpha^* > a(x, t) > \alpha > 0, & x > \xi.
  \end{cases}
  \]

- Since the equation in (1) is of convection-diffusion type in \( G^+ \) and the convection coefficient is positive in \( \Omega^+ \), an interior layer of width \( O(\varepsilon) \) appears on the left side of the boundary of \( G^+ \).

- Whereas the equation in (1) is of parabolic reaction-diffusion type in \( G^- \), which leads to occurrence of parabolic boundary layers of width \( O(\sqrt{\varepsilon}) \) on both the boundaries of \( G^- \).

- **Application.** Such kind of problems describe, for example, an electromagnetic field arising in the motion of a train on an air-pillow.

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Consider the following parabolic IBVP:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + x(1 - x)u &= 2(1 + x^2)t^2, \quad (x, t) \in (0, 0.5) \times (0, 1], \\
-\varepsilon \frac{\partial^2 u}{\partial x^2} - (1 + x(1 - x))\frac{\partial u}{\partial x} + x(1 - x)u &= 3(1 + x^2)t^2, \quad (x, t) \in (0.5, 1) \times (0, 1], \\
u(x, 0) &= 0, \quad 0 \leq x \leq 1, \\
u(0, t) &= t^2, \quad u(1, t) = 0, \quad 0 < t \leq 1.
\end{aligned}
\]
Surface Plot of Numerical Soln. for $\varepsilon = 1e^{-4}, N = 64$
Objective

To construct and analyze an $\varepsilon$-\textit{uniformly convergent} hybrid numerical scheme for singularly perturbed mixed parabolic-elliptic type problems (1), exhibiting both the \textit{boundary} and the \textit{interior layers}.
Objective

To construct and analyze an $\varepsilon$-uniformly convergent hybrid numerical scheme for singularly perturbed mixed parabolic-elliptic type problems (1), exhibiting both the boundary and the interior layers.

Definition

$\varepsilon$-uniformly convergent numerical method: A numerical method is said to be $\varepsilon$-uniformly convergent if

$$
\sup_{0<\varepsilon\leq 1} \| u - U \| \leq C((\Delta t)^p + N^{-r}),
$$

where $C$ is independent of mesh points, mesh size and the parameter $\varepsilon$.

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Properties of Analytical Solution

Here, we study the behavior of the continuous solution and the bounds of its derivatives.
Let $\Gamma = \overline{G \setminus G}$.
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Let $\Gamma = \overline{G} \setminus G$.

**Lemma**

**Maximum Principle.** Suppose that a function $g \in C^0(\overline{G}) \cap C^2(G^- \cup G^+)$ satisfies $g(x, t) \leq 0$, $(x, t) \in \Gamma$, $\left[ \frac{\partial g}{\partial x} \right](\xi, t) \geq 0$, $t > 0$ and

$$
\begin{cases}
L_1, \varepsilon g(x, t) \leq 0, \ (x, t) \in G^- , \\
L_2, \varepsilon g(x, t) \leq 0, \ (x, t) \in G^+ ,
\end{cases}
$$

then $g(x, t) \leq 0$, $\forall (x, t) \in \overline{G}$.

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An immediate consequence of the above maximum principle is the following stability result, which implies the uniqueness of the solution of the IBVP (1)-(2).
An immediate consequence of the above maximum principle is the following stability result, which implies the uniqueness of the solution of the IBVP (1)-(2).

\[ \| u \|_{\infty, \mathcal{G}} \leq \| u \|_{\infty, \Gamma} + \frac{(1 + T)}{\mu} \| f \|_{\infty, \mathcal{G}}, \]

where \( \mu = \min \{ 1, \alpha/(1 - \xi) \} \).
Consider the decomposition of the solution $u$ as $u = v + w$, where $v, w$ are respectively the smooth and interior layer components.
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The smooth component $v$ is defined to be the sum

$$v = \sum_{i=0}^{3} \varepsilon^i v_i + \varepsilon^4 \mathcal{A}, \quad v_i = \begin{cases} v_i^- & \text{in } G^-, \\ v_i^+ & \text{in } G^+. \end{cases}$$

(4)
Consider the decomposition of the solution $u$ as $u = v + w$, where $v, w$ are respectively the smooth and interior layer components.

The smooth component $v$ is defined to be the sum

$$v = \sum_{i=0}^{3} \varepsilon^i v_i + \varepsilon^4 \mathcal{R}, \quad v_i = \begin{cases} v_i^- & \text{in } G^-, \\ v_i^+ & \text{in } G^+. \end{cases}$$

Here, the functions $v_i^-, i = 0, 1, 2, 3$ are solutions to the following first-order problems

$$
\begin{cases} 
\frac{\partial v_0^-}{\partial t} + b v_0^- = f & \text{in } G^-, \quad v_0^-(x, 0) = u(x, 0), \quad x \in \overline{\Omega}^-,
\\
\frac{\partial v_i^-}{\partial t} + b v_i^- = \frac{\partial^2 v_{i-1}^-}{\partial x^2} & \text{in } G^-, \quad v_i^-(x, 0) = 0, \quad x \in \overline{\Omega}^-, \quad i = 1, 2, 3.
\end{cases}
$$

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Solution Decomposition

And the functions $v_i^+, \ i = 0, 1, 2, 3$ satisfy the following first-order problems

\[
\begin{align*}
- a \frac{\partial v_0^+}{\partial x} + b v_0^+ &= f \quad \text{in } G^+, \\
v_0^+(1, t) &= u(1, t), \quad t \in (0, T], \quad v_0^+(x, 0) = u(x, 0), \quad x \in \overline{\Omega}^+, \\
- a \frac{\partial v_i^+}{\partial x} + b v_i^+ &= \frac{\partial^2 v_{i-1}^+}{\partial x^2} \quad \text{in } G^+, \\
v_i^+(1, t) &= 0, \quad t \in (0, T], \quad v_i^+(x, 0) = 0, \quad x \in \overline{\Omega}^+, \quad i = 1, 2, 3.
\end{align*}
\]

(6)
Solution Decomposition

And the functions $v_i^+, i = 0, 1, 2, 3$ satisfy the following first-order problems

\[
\begin{align*}
-a \frac{\partial v_0^+}{\partial x} + b v_0^+ &= f \quad \text{in } G^+, \\
v_0^+(1, t) &= u(1, t), \quad t \in (0, T], \quad v_0^+(x, 0) = u(x, 0), \quad x \in \overline{\Omega}^+,
\end{align*}
\]

\[
\begin{align*}
-a \frac{\partial v_i^+}{\partial x} + b v_i^+ &= \frac{\partial^2 v_{i-1}^+}{\partial x^2} \quad \text{in } G^+, \\
v_i^+(1, t) &= 0, \quad t \in (0, T], \quad v_i^+(x, 0) = 0, \quad x \in \overline{\Omega}^+, \quad i = 1, 2, 3.
\end{align*}
\]

And lastly, the remainder $\mathcal{R}$ satisfies

\[
\begin{align*}
L_{1,\varepsilon} \mathcal{R} &= \frac{\partial^2 v_3^-}{\partial x^2} \quad \text{in } G^-, \quad L_{2,\varepsilon} \mathcal{R} = \frac{\partial^2 v_3^+}{\partial x^2} \quad \text{in } G^+, \\
\mathcal{R}(0, t) &= \mathcal{R}(1, t) = 0, \quad \mathcal{R}(x, 0) = 0, \quad x \in \overline{\Omega}, \\
[\mathcal{R}](\xi, t) &= 0, \quad \left[ \frac{\partial \mathcal{R}}{\partial x} \right](\xi, t) = 0, \quad t \in (0, T].
\end{align*}
\]

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Solution Decomposition

The layer component $w$ is further decomposed into the sum

$$w = \begin{cases} 
  w^1 + w^2 & \text{in } G^-, \\
  w^2 & \text{in } G^+,
\end{cases}$$

(8)
Solution Decomposition

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\[ w = \begin{cases} 
  w^1 + w^2 & \text{in } G^-, \\
  w^2 & \text{in } G^+, 
\end{cases} \quad (8) \]

where $w^1$ and $w^2$ respectively satisfy the following IBVPs:

\[
\begin{cases} 
  L_{1,\varepsilon}w^1 = 0 & \text{in } G^-, \\
  w^1(0, t) = u(0, t) - v(0, t), & w^1(\xi, t) = 0, \quad t \in (0, T], \\
  w^1(x, 0) = 0, & x \in \overline{\Omega}^-,
\end{cases}
\quad (9)
\]
Solution Decomposition

The layer component $w$ is further decomposed into the sum

$$
w = \begin{cases} 
  w^1 + w^2 & \text{in } G^-, \\
  w^2 & \text{in } G^+, 
\end{cases} \quad (8)
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where $w^1$ and $w^2$ respectively satisfy the following IBVPs:

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\begin{align*}
L_{1,\varepsilon} w^1 &= 0 \quad \text{in } G^-, \\
w^1(0, t) &= u(0, t) - v(0, t), \\
w^1(\xi, t) &= 0, \quad t \in (0, T], \\
w^1(x, 0) &= 0, \quad x \in \overline{\Omega}^-,
\end{align*} \quad (9)
$$

and

$$
\begin{align*}
L_{1,\varepsilon} w^2 &= 0 \quad \text{in } G^-, \\
L_{2,\varepsilon} w^2 &= 0 \quad \text{in } G^+, \\
w^2(0, t) &= w^2(1, t) = 0, \\
w^2(x, 0) &= 0, \quad x \in \overline{\Omega}, \\
[w^2](\xi, t) &= -[v](\xi, t), \\
\left[ \frac{\partial w^2}{\partial x} \right](\xi, t) &= - \left[ \frac{\partial v}{\partial x} \right](\xi, t) + \frac{\partial w^1}{\partial x}(\xi, t), \quad t \in (0, T)
\end{align*}
$$

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Apriori Bounds on the Smooth & Singular Components

**Theorem**

For all integers $l, m$, satisfying $0 \leq l + 2m \leq 4$, the smooth component $v$ satisfies the bounds

$$
\left\| \frac{\partial^{l+m} v}{\partial x^l \partial t^m} \right\|_{\infty, G^- \cup G^+} \leq C,
$$

and the layer components $w^1$ and $w^2$, respectively, satisfy the bounds

$$
\left| \frac{\partial^{l+m} w^1}{\partial x^l \partial t^m}(x, t) \right| \leq C \left( \varepsilon^{-1/2} \exp \left( -x/\sqrt{\varepsilon} \right) \right), \quad (x, t) \in G^-,
$$

$$
\left| \frac{\partial^{l+m} w^2}{\partial x^l \partial t^m}(x, t) \right| \leq \begin{cases} 
C \left( \varepsilon^{-1/2} \exp \left( -\left(\xi - x\right)/\sqrt{\varepsilon} \right) \right), & (x, t) \in G^-, \\
C \left( \varepsilon^{-1} \exp \left( -\left( x - \xi \right)\alpha/\varepsilon \right) \right), & (x, t) \in G^+. 
\end{cases}
$$

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Let $G_{\varepsilon}^{N,M} = \Omega_{x}^{N,\varepsilon} \times S_{t}^{M}$ be the rectangular mesh on $G = \overline{\Omega} \times [0, T]$. 
Let $\overline{G}^{N,M}_{\varepsilon} = \overline{\Omega}^{N,M}_{x,\varepsilon} \times S^M_t$ be the rectangular mesh on $\overline{G} = \overline{\Omega} \times [0, T]$. 

$S^M_t$ is a uniform mesh with $M$ mesh-intervals on $[0, T]$. 

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Discretization of Domain

Let $\overline{G}_\varepsilon^{N,M} = \overline{\Omega}_x^N \times S_t^M$ be the rectangular mesh on $\overline{G} = \overline{\Omega} \times [0, T]$.

$S_t^M$ is a uniform mesh with $M$ mesh-intervals on $[0, T]$.

To define a piecewise-uniform Shishkin mesh $\overline{\Omega}_x^{N,\varepsilon}$, the spatial domain $\overline{\Omega}$ is divided into five subintervals as

$$\overline{\Omega} = [0, \sigma_1] \cup [\sigma_1, \xi - \sigma_1] \cup [\xi - \sigma_1, \xi] \cup [\xi, \xi + \sigma_2] \cup [\xi + \sigma_2, 1].$$
Choose $\sigma_1 = \min\left\{ \frac{\xi}{4}, \tau_1 \sqrt{\varepsilon \ln N} \right\}$, \hspace{10pt} \sigma_2 = \min\left\{ \frac{1 - \xi}{2}, \frac{2\tau_2 \varepsilon}{\alpha \ln N} \right\}$. 
Choose $\sigma_1 = \min \left\{ \frac{\xi}{4}, \tau_1 \sqrt{\varepsilon \ln N} \right\}$, \quad $\sigma_2 = \min \left\{ \frac{1 - \xi}{2}, \frac{2 \tau_2 \varepsilon}{\alpha} \ln N \right\}$.

- $N/8$ mesh-intervals in $[0, \sigma_1], [\xi - \sigma_1, \xi]$. 

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Choose $\sigma_1 = \min\left\{ \frac{\xi}{4}, \tau_1 \sqrt{\varepsilon \ln N} \right\}$, \quad $\sigma_2 = \min\left\{ \frac{1 - \xi}{2}, \frac{2\tau_2 \varepsilon}{\alpha} \ln N \right\}$.

- $N/8$ mesh-intervals in $[0, \sigma_1], [\xi - \sigma_1, \xi]$.
- $N/4$ mesh-intervals in $[\sigma_1, \xi - \sigma_1], [\xi, \xi + \sigma_2], [\xi + \sigma_2, 1]$. 

Natesan Srinivasan,
Discretization of Domain

Choose \( \sigma_1 = \min \left\{ \frac{\xi}{4}, \tau_1 \sqrt{\varepsilon \ln N} \right\} \), \( \sigma_2 = \min \left\{ \frac{1 - \xi}{2}, \frac{2\tau_2 \varepsilon}{\alpha} \ln N \right\} \).

- \( N/8 \) mesh-intervals in \([0, \sigma_1], [\xi - \sigma_1, \xi] \).
- \( N/4 \) mesh-intervals in \([\sigma_1, \xi - \sigma_1], [\xi, \xi + \sigma_2], [\xi + \sigma_2, 1] \).

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The proposed hybrid numerical scheme takes the following form:

\[
\begin{align*}
L_{1,\text{cen}}^{N,M,(-)} U_{i}^{n+1} &= f_{i}^{n+1}, \\
L_{2,\text{cen}}^{N,M,(+)} U_{i}^{n+1} &= f_{i}^{n+1}, \\
L_{2,mu}^{N,M} U_{i}^{n+1} &= f_{i+1/2}^{n+1}, \\
D_{x}^{F} U_{i}^{n+1} - D_{x}^{B} U_{i}^{n+1} &= 0,
\end{align*}
\]

for \( i = 1, \ldots, N/2 - 1 \), for \( i = N/2 + 1, \ldots, 3N/4 - 1 \), for \( i = 3N/4, \ldots, N - 1 \), for \( i = N/2 \).

(11)

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The proposed **hybrid numerical scheme** takes the following form:

\[
\begin{align*}
L_{1,\text{cen}}^{N,M,(-)} U_i^{n+1} & = f_i^{n+1}, \quad \text{for } i = 1, \ldots, N/2 - 1, \\
L_{2,\text{cen}}^{N,M,(+)} U_i^{n+1} & = f_i^{n+1}, \quad \text{for } i = N/2 + 1, \ldots, 3N/4 - 1, \\
L_{2,\text{mu}}^{N,M} U_i^{n+1} & = f_{i+1/2}^{n+1}, \quad \text{for } i = 3N/4, \ldots, N - 1, \\
D_F^x U_i^{n+1} - D_B^x U_i^{n+1} & = 0, \quad \text{for } i = N/2.
\end{align*}
\]

(11)

Rearranging the terms in (11), we obtain

\[
\begin{align*}
U_i^0 & = s_0(x_i), \quad \text{for } i = 0, \ldots, N, \\
L_{\varepsilon}^{N,M} U_i^{n+1} & = \tilde{f}_i^{n+1}, \quad \text{for } i = 1, \ldots, N - 1, \\
U_0^{n+1} & = s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \\
\text{for } n = 0, \ldots, M - 1.
\end{align*}
\]

(12)
The finite difference operator $L_{\varepsilon}^{N,M}$ is defined as

$$L_{\varepsilon}^{N,M} U_{i}^{n+1} = \begin{cases} \left[ r_i^{-} U_{i-1}^{n+1} + r_i^{0} U_{i}^{n+1} + r_i^{+} U_{i+1}^{n+1} \right] + \left[ p_i^{-} U_{i-1}^{n} + p_i^{0} U_{i}^{n} + p_i^{+} U_{i+1}^{n} \right], \\ & \text{for } i = 1, \ldots, N/2 - 1, N/2 + 1, \ldots, N - 1, \\ q_i^{-,2} U_{i-2}^{n+1} + q_i^{-,1} U_{i-1}^{n+1} + q_i^{0} U_{i}^{n+1} + q_i^{+,1} U_{i+1}^{n+1} + q_i^{+,2} U_{i+2}^{n+1}, \\ & \text{for } i = N/2, \end{cases}$$

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The finite difference operator $L_{\varepsilon,M}^{N,M}$ is defined as

$$L_{\varepsilon,M}^{N,M} U_i^{n+1} = \begin{cases} 
[r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}] + [p_i^- U_{i-1}^n + p_i^0 U_i^n + p_i^+ U_{i+1}^n], & \text{for } i = 1, \ldots, N/2 - 1, N/2 + 1, \ldots, N - 1, \\
q_i^{-,2} U_{i-2}^{n+1} + q_i^{-,1} U_{i-1}^{n+1} + q_i^0 U_i^{n+1} + q_i^{+,1} U_{i+1}^{n+1} + q_i^{+,2} U_{i+2}^{n+1}, & \text{for } i = N/2,
\end{cases}$$

where

$$q_i^{-,2} = -\frac{1}{2h(l)}, \quad q_i^{-,1} = \frac{2}{h(l)}, \quad q_i^0 = -\frac{3}{2 \left( \frac{1}{h(r)} + \frac{1}{h(l)} \right)}, \quad q_i^{+,1} = \frac{2}{h(r)}, \quad q_i^{+,2} = -\frac{1}{2h(r)}.$$
Finite Difference Scheme

The finite difference operator $L_{\varepsilon}^{N, M}$ is defined as

$$L_{\varepsilon}^{N, M} U_{i}^{n+1} = \begin{cases} 
[r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}] + [p_i^- U_{i-1}^n + p_i^0 U_i^n + p_i^+ U_{i+1}^n], \\
for i = 1, \ldots, N/2 - 1, \ N/2 + 1, \ldots, N - 1, \\
q_i^- U_{i-2}^{n+1} + q_i^- U_{i-1}^{n+1} + q_i^0 U_i^{n+1} + q_i^+ U_{i+1}^{n+1} + q_i^+ U_{i+2}^{n+1}, \\
for i = N/2,
\end{cases}$$

where

$$q_i^- = -\frac{1}{2h(l)}, \ q_i^- = \frac{2}{h(l)}, \ q_i^0 = -\frac{3}{2} \left( \frac{1}{h(r)} + \frac{1}{h(l)} \right), \ q_i^+ = \frac{2}{h(r)}, \ q_i^+ = -\frac{1}{2h(r)}.$$

The right hand side vector $\tilde{f}^{n+1}$ as

$$\tilde{f}_{i}^{n+1} = \begin{cases} 
[m_i f_{i-1}^{n+1} + m_i f_i^{n+1} + m_i f_{i+1}^{n+1}], \\
for i = 1, \ldots, N/2 - 1, \ N/2 + 1, \ldots, N - 1, \\
0, \ \text{for } i = N/2.
\end{cases}$$

Natesan Srinivasan,
The difference operator $L_{ε}^{N,M}$ does not satisfy the discrete minimum principle.
The difference operator $L_{N,M}^{N,M}$ does not satisfy the **discrete minimum principle**.

We transform the difference scheme (12) into the following form:

\[
\begin{align*}
U_i^0 &= s_0(x_i), \quad \text{for } i = 0, \ldots, N, \\
L_{H}^{N,M} U_i^{n+1} &= \tilde{f}_{H,i}^{n+1}, \quad \text{for } i = 1, \ldots, N - 1, \\
U_0^{n+1} &= s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \\
\text{for } n = 0, \ldots, M - 1.
\end{align*}
\]

(13)
The difference operator $L^{N,M}_H$ and the right hand side term $\tilde{f}^{n+1}_{H,i}$ are respectively defined as

\[
\begin{align*}
L^{N,M}_H U^{n+1}_i &= \left[ r_i^- U^{n+1}_{i-1} + r_i^0 U^{n+1}_i + r_i^+ U^{n+1}_{i+1} \right] + \left[ p_i^- U^n_{i-1} + p_i^0 U^n_i + p_i^+ U^n_{i+1} \right], \\
\tilde{f}^{n+1}_{H,i} &= \left[ m_i^- f^{n+1}_{i-1} + m_i^0 f^{n+1}_i + m_i^+ f^{n+1}_{i+1} \right].
\end{align*}
\]
Finite Difference Scheme

The difference operator \( L_{H}^{N,M} \) and the right hand side term \( \tilde{f}_{H,i}^{n+1} \) are respectively defined as

\[
\begin{align*}
L_{H}^{N,M} U_{i}^{n+1} &= \left[ r_{i}^{-} U_{i-1}^{n+1} + r_{i}^{0} U_{i}^{n+1} + r_{i}^{+} U_{i+1}^{n+1} \right] + \left[ p_{i}^{-} U_{i-1}^{n} + p_{i}^{0} U_{i}^{n} + p_{i}^{+} U_{i+1}^{n} \right], \\
\tilde{f}_{H,i}^{n+1} &= \left[ m_{i}^{-} f_{i-1}^{n+1} + m_{i}^{0} f_{i}^{n+1} + m_{i}^{+} f_{i+1}^{n+1} \right].
\end{align*}
\]

Here, for \( i = N/2 \),

\[
\begin{align*}
r_{i}^{-} &= -\frac{1}{2h_{l}} \left[ 4 - \frac{2(2\varepsilon + h_{l}^{2} b_{i-1}^{n+1} + \frac{h_{l}^{2}}{\Delta t})}{2\varepsilon} \right], \\
r_{i}^{0} &= \frac{1}{2h_{r}} \left( 3 - \frac{2\varepsilon - h_{r} a_{i}^{n+1}}{2\varepsilon + h_{r} a_{i}^{n+1}} \right) + \frac{1}{h_{l}}, \\
r_{i}^{+} &= -\frac{1}{2h_{r}} \left[ 4 - \frac{2(2\varepsilon + h_{r} b_{i+1}^{n+1})}{2\varepsilon + h_{r} a_{i+1}^{n+1}} \right], \\
p_{i}^{-} &= -\frac{h_{l}}{2\varepsilon \Delta t}, \quad p_{i}^{0} = 0, \quad p_{i}^{+} = 0, \\
m_{i}^{-} &= \frac{h_{l}}{2\varepsilon}, \quad m_{i}^{0} = 0, \quad m_{i}^{+} = \frac{h_{r}}{2\varepsilon + h_{r} a_{i+1}^{n+1}}.
\end{align*}
\]

Natesan Srinivasan,
Let $G^{N,M} = \overline{G^{N,M}} \cap G$ and $\Gamma^{N,M} = \overline{G^{N,M}} \setminus G^{N,M}$.

Lemma

Assume that $N \geq N_0$, where

$$\frac{N_0}{\ln N_0} \geq 4\tau_2 \frac{\alpha^*}{\alpha},$$

$$\frac{\alpha N_0}{2} \geq \|b\|_\infty \quad \text{and} \quad (\|b\|_\infty + \Delta t^{-1}) \leq \frac{2kN_0^2}{\ln^2 N_0},$$

where $k = \left(\frac{\eta}{8\tau_1}\right)^2$. Then, the difference operator $L_{H}^{N,M}$ satisfies a discrete minimum principle, i.e., if the mesh function $Z$ satisfies $Z \geq 0$ on $\Gamma^{N,M}$, then $L_{H}^{N,M}Z \geq 0$ in $G^{N,M}$ implies that $Z \geq 0$ at each point of $G^{N,M}$.
Consequently, we obtain the following stability estimate.

**Lemma**

Let $U$ be the solution of (13) and let the assumptions (14) and (15) hold true. Then,

$$
\| U \|_{\infty, G_{\varepsilon}^{N,M}} \leq \| U \|_{\infty, \Gamma_{\varepsilon}^{N,M}} + \frac{(1 + T)}{\mu} \| \tilde{f}_H \|_{\infty, \overline{G}_{\varepsilon}^{N,M}},
$$

where $\mu = \min \{ 1, \alpha/(1 - \xi) \}$.
Outline

1. Model Problem
2. Properties of Analytical Solution
3. The Numerical Scheme
4. Error Analysis
5. Numerical Results
6. Conclusions

Natesan Srinivasan,
Decomposition of Discrete Solution

As like the continuous solution, we decompose the discrete solution $U$ into the following sum:

$$U^{n+1}_i = \begin{cases} 
V^{n+1}_{L,i} + W^{n+1}_{L,i}, & i = 1, \ldots, N/2 - 1, \\
V^{n+1}_{L,i} + W^{n+1}_{L,i} = V^{n+1}_{R,i} + W^{n+1}_{R,i}, & i = N/2, \\
V^{n+1}_{R,i} + W^{n+1}_{R,i}, & i = N/2 + 1, \ldots, N - 1. 
\end{cases}$$

Natesan Srinivasan,
Decomposition of Discrete Solution

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V^{n+1}_{R,i} + W^{n+1}_{R,i}, & i = N/2 + 1, \ldots, N - 1.
\end{cases}$$

The mesh functions $V_L$ and $V_R$ approximate $\nu$ respectively to the left and the right of the point of discontinuity $x = \xi$.

*Natesan Srinivasan,*
Decomposition of Discrete Solution

As like the continuous solution, we **decompose** the **discrete** solution $U$ into the following sum:

$$U_{i}^{n+1} = \begin{cases} 
 V_{L,i}^{n+1} + W_{L,i}^{n+1}, & i = 1, \ldots, N/2 - 1, \\
 V_{L,i}^{n+1} + W_{L,i}^{n+1} = V_{R,i}^{n+1} + W_{R,i}^{n+1}, & i = N/2, \\
 V_{R,i}^{n+1} + W_{R,i}^{n+1}, & i = N/2 + 1, \ldots, N - 1. 
\end{cases}$$

The mesh functions $V_L$ and $V_R$ approximate $v$ respectively to the left and the right of the point of discontinuity $x = \xi$.

We construct the mesh functions $W_L$ and $W_R$ such that

$$W_L = W^1 + W_L^2 \quad \text{and} \quad W_R = W_R^2,$$

where $W^1$ approximates $w^1$ and $W_L^2$, $W_R^2$ approximates $w^2$ respectively on either side of $x = \xi$.

Natesan Srinivasan,
Error Analysis

First, the error $|U_{i}^{n+1} - u(x_i, t_{n+1})|$ is analyzed in the region $G^- \cup G^+$ by separately estimating the error for the smooth components $V_L$ and $V_R$ of the solution $U$.

Natesan Srinivasan,
Error Analysis

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Lemma

Assume that $\varepsilon \leq N^{-1}$. Then under the assumptions (14) and (15), the errors associated to the smooth components satisfy the following estimates

\[
\begin{align*}
|V_{L,i}^{n+1} - v(x_i, t_{n+1})| &\leq C \left( N^{-2} + \Delta t \right), & \text{for } 1 \leq i \leq N/2 - 1, \\
|V_{R,i}^{n+1} - v(x_i, t_{n+1})| &\leq C N^{-2}, & \text{for } N/2 + 1 \leq i \leq N - 1.
\end{align*}
\]

Natesan Srinivasan,
Error Analysis

Now, for further error analysis we define the mesh function

\[ Q_i = \prod_{j=1}^{N-i} \left( 1 + \frac{\alpha \bar{h}_j}{2\varepsilon} \right), \quad \text{for } i = N/2, \ldots, N, \]

(with the usual convention that if \( i = N \), then \( Q_N = 1 \)) where

\[ \bar{h}_j = \begin{cases} H(r), & \text{for } 1 \leq j \leq N/4, \\ h(r), & \text{for } N/4 + 1 \leq j \leq N/2. \end{cases} \]

Natesan Srinivasan,
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Here, the mesh function \( \left\{ Q_i/Q_{N/2} \right\} \left( = \prod_{j=N-i+1}^{N/2} \left( 1 + \frac{\alpha \bar{h}_j}{2\varepsilon} \right)^{-1} \right) \) is a discrete analogue to the function \( \exp \left( - \left( x_i - \xi \right) \alpha / 2\varepsilon \right) \).

Natesan Srinivasan,
Error Analysis

Next, we analyze the errors for the layer components $W_L$ and $W_R$ of the solution $U$ in the region $((0, \xi - \sigma_1] \cup [\xi + \sigma_2, 1)) \times (0, T]$. 

Natesan Srinivasan,
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Next, we analyze the errors for the layer components $W_L$ and $W_R$ of the solution $U$ in the region $((0, \xi - \sigma_1] \cup [\xi + \sigma_2, 1)) \times (0, T]$.

Lemma

Let $\tau_1, \tau_2 \geq 2$. Then under the assumptions (14) and (15), the errors associated to the layer components satisfy the following estimates

$$
\left| W_{L,i}^{n+1} - w(x_i, t_{n+1}) \right| \leq \begin{cases} 
C \left( N^{-2} \ln^2 N + \Delta t \right), & \text{for } 1 \leq i \leq N/8 - 1, \\
C \left( N^{-2} + \Delta t \right), & \text{for } N/8 \leq i \leq 3N/8,
\end{cases}
$$

and

$$
\left| W_{R,i}^{n+1} - w(x_i, t_{n+1}) \right| \leq CN^{-2}, \quad \text{for } 3N/4 \leq i \leq N - 1.
$$

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Main Convergence Result

The following theorem provides the almost second-order $\varepsilon$–uniform convergence of the numerical solution.
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**Theorem**

Assume that $N \geq N_0$ satisfies the conditions given in (14), (15) and $\varepsilon \leq N^{-1}$. Then if $\tau_1, \tau_2 \geq 2$, the respective solutions $u$ and $U$ of (1)-(2) and (13) satisfy the following error bounds at time level $t_n$:

$$
|U_i^n - u(x_i, t_n)| \leq \begin{cases} 
C \left[ N^{-2} \ln^2 N + \Delta t \right], & \text{for } 1 \leq i < N/8, \\
C \left[ N^{-2} + \Delta t \right], & \text{for } N/8 \leq i \leq 3N/8, \\
C \left[ N^{-2} \ln^3 N + N^{-1} \ln^2 N \Delta t + \Delta t \right], & \text{for } 3N/8 < i < 3N/4, \\
CN^{-2}, & \text{for } 3N/4 \leq i \leq N - 1.
\end{cases}
$$

Natesan Srinivasan,
Outline

1 Model Problem

2 Properties of Analytical Solution

3 The Numerical Scheme

4 Error Analysis

5 Numerical Results

6 Conclusions
Example

Consider the following parabolic IBVP:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + x(1 - x)u &= 2(1 + x^2)t^2, \quad (x, t) \in (0, 0.5) \times (0, 1], \\
-\varepsilon \frac{\partial^2 u}{\partial x^2} - (1 + x(1 - x)) \frac{\partial u}{\partial x} + x(1 - x)u &= 3(1 + x^2)t^2, \\
\end{aligned}
\]

\[
\begin{aligned}
(x, t) &\in (0.5, 1) \times (0, 1], \\
u(x, 0) &= 0, \quad 0 \leq x \leq 1, \\
u(0, t) &= t^2, \quad u(1, t) = 0, \quad 0 < t \leq 1.
\end{aligned}
\]
Surface Plot of Numerical Soln. for $\varepsilon = 1e - 4, N = 64$

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Behaviour of Numerical Soln. as $\varepsilon \to 0$

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Loglog Plot of Max. point-wise Errors

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### Max. point-wise Errors (Spatial Effect)

<table>
<thead>
<tr>
<th>N</th>
<th>boundary layer region</th>
<th>left outer region</th>
<th>interior layer region</th>
<th>right outer region</th>
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<td>[$\sigma_1, \xi - \sigma_1$]</td>
<td>($\xi - \sigma_1, \xi + \sigma_2$)</td>
<td>[$\xi + \sigma_2, 1$]</td>
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<td>1.9198</td>
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Max. point-wise Errors (Spatial Effect)

<table>
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<tr>
<th>N</th>
<th>$\varepsilon = 10^{-7}$</th>
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<td></td>
<td>boundary layer region</td>
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<tr>
<td></td>
<td>$[0, \sigma_1)$</td>
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<tr>
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<tr>
<td>256</td>
<td>6.9840e-4 8.0643e-6 3.1960e-3 2.3782e-5</td>
</tr>
</tbody>
</table>

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Outline

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Conclusions

An efficient numerical method is proposed and analyzed on a layer-resolving Shishkin mesh for a class of singularly perturbed IBVP (1)-(2) of mixed parabolic-elliptic type.
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The proposed hybrid scheme has the following properties:
Conclusions

An efficient numerical method is proposed and analyzed on a layer-resolving Shishkin mesh for a class of singularly perturbed IBVP (1)-(2) of mixed parabolic-elliptic type.

The proposed hybrid scheme has the following properties:

- $\varepsilon$-uniformly stable
Conclusions

An efficient numerical method is proposed and analyzed on a layer-resolving Shishkin mesh for a class of singularly perturbed IBVP (1)-(2) of mixed parabolic-elliptic type.

The proposed hybrid scheme has the following properties:

- $\epsilon$-uniformly stable
- $\epsilon$-uniformly convergent
Conclusions

An efficient numerical method is proposed and analyzed on a layer-resolving Shishkin mesh for a class of singularly perturbed IBVP (1)-(2) of mixed parabolic-elliptic type.

The proposed hybrid scheme has the following properties:

- $\varepsilon$-uniformly stable
- $\varepsilon$-uniformly convergent
- temporal order of convergence is one

Natesan Srinivasan,
Conclusions

An efficient numerical method is proposed and analyzed on a layer-resolving Shishkin mesh for a class of singularly perturbed IBVP (1)-(2) of mixed parabolic-elliptic type.

The proposed hybrid scheme has the following properties:

- $\epsilon$-uniformly stable
- $\epsilon$-uniformly convergent
- temporal order of convergence is one
- spatial order of convergence is almost two

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I.A. Braianov.
Numerical solution of a mixed singularly perturbed parabolic-elliptic problem.
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I.A. Braianov.
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Thank You