An introduction to Finite Difference methods for PDEs in Finance

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1 Introduction

In this lecture, I discuss the practical aspects of designing Finite Difference methods for Hamilton-Jacobi-Bellman equations of parabolic type arising in Quantitative Finance. The approach is based on the very powerful and simple framework developed by Barles-Souganidis [3]. They prove very elegantly, using viscosity solutions techniques, the convergence of any consistent, monotone and stable approximation scheme. The key property here is the monotonicity which guarantees that the scheme satisfies the same Ellipticity condition as the HJB operator. I will provide a number of examples of monotone schemes in these notes. In practice, pure Finite Difference schemes are only useful in 1,2 or at most 3 spatial dimensions. One of their merits is to be quite simple and easy to implement. They can also be combined with Monte Carlo methods to solve nonlinear parabolic PDEs (see [5]).

Such approximations are now fairly standard and you will find many interesting examples available in the literature. For instance, I suggest the articles on the subject by P. Forsyth (see [8], [6], [9] for instance). Finally, for a basic introduction to Finite Difference methods for linear parabolic PDEs, I recommend the book by J.W. Thomas [7].

2 Quick overview of the Barles-Souganidis framework [3]

Consider the parabolic PDE

$$u_t + F(t, x, u, Du, D^2u) = 0 \text{ in } (0, T] \times IR^N \quad (1)$$

$$u(0, x) = u_0(x) \text{ in } IR^N \quad (2)$$

where F is Elliptic

$$F(x,u,p,A) \leq F(x,u,p,B), \text{ if } A \geq B.$$

For a sake of simplicity, we assume that u_0 is bounded in IR^N. Furthermore, we assume that (1), (2) satisfy a strong comparison principle:

Comparison Principle

Let u be any bounded usc vicosity subsolution of (1), v be any bounded lsc vicosity supersolution of (1) such that $u(0,x) \leq v(0,x)$, then

$$u \leq v.$$

The main application we have in mind is to an operator F coming from a standard stochastic control problem:

$$F(t,x,r,p,X) = \inf_{\alpha \in \mathcal{A}} \{-tr[a^\alpha(t,x)X] - b^\alpha(t,x)p - c^\alpha(t,x)r - f^\alpha(t,x)\}$$

where $a^\alpha = \frac{1}{2}\sigma^\alpha \sigma^{\alpha T}$.

Typically, the set of control $\mathcal{A}$ is compact or finite, all the coefficients in the equations are bounded and Lipschitz continuous in x, Hölder with coefficient $\frac{1}{2}$ in t and all the bounds are independent of α. Then the unique viscosity solution u of (1) is a bounded and Lipschitz continuous function and is the solution of the underlying stochastic control problem. The ideas, concepts and techniques actually apply to a broader range of optimal control problems. In particular, you can adapt the techniques to handle different situations, even possibly treat some delicate singular control problems.

The aim is to build an approximation scheme which preserves the Ellipticity. This discrete Ellipticity property is called **monotonicity**. The monotonicity, together with the consistency of the scheme and some regularity ensure its convergence to the unique viscosity solution of the PDE (1),(2). It is worth insisting on the fact that if the scheme is not monotone, it may fail to converge to the correct solution (see [6] for an example)!

A numerical scheme is an equation of the following form

$$S(h,t,x,u_h(t,x),[u_h]_{t,x}) = 0 \text{ for } (t,x) \text{ in } \mathcal{G}_h \backslash \{t=0\} \tag{3}$$

$$u_h(0,x) = u_{h,0}(x) \text{ in } \mathcal{G}_h \cap \{t=0\} \tag{4}$$

where $h = (\Delta t, \Delta x)$, $\mathcal{G}_h = \Delta t\{0,1,...,n_T\} \times \Delta x Z^N$,$u_h$ stands for the approximation of u and $[u_h]_{t,x}$ represents the value of u_h at other points than (t,x). The theory requires the following assumptions:

Monotonicity If $u \leq v$,

$$S(h, t, x, r, u) \geq S(h, t, x, r, v)$$

Consistency

For every smooth function $\phi(t, x)$,

$$S(h, t, x, \Phi(t, x), [\Phi(t, x)]_{t,x}) \rightarrow_{h \rightarrow 0} \Phi_t + F(t, x, \Phi(t, x), D\Phi, D^2\Phi).$$

Stability

For every $h > 0$, the scheme has a solution u_h which is uniformly bounded independently of h.

Theorem (Barles-Souganidis[3])
Under the above assumptions, if the scheme (3),(4) satisfy the consistency, monotonicity and stability property, its solution u_h converges locally uniformly to the unique viscosity solution of (1),(2).

3 First examples

3.1 The heat equation: the classic explicit and implicit schemes

First, let me recall the classic explicit and implicit schemes for the heat equation and verify that these schemes satisfy the required properties.

$$u_t - u_{xx} = 0 \text{ in } (0, T] \times IR. \tag{5}$$

$$u(0, x) = u_0(x) \tag{6}$$

Next consider the well-known linear heat equation whose treatment does not require the machinery of viscosity solutions but falls into the scope of this theory and provides the opportunity to understand the connection between the theory for linear parabolic equations and the theory of viscosity solutions. More precisely, our goal here is to verify that the standard finite difference approximations for the heat equation are convergent in the Barles-Souganidis sense.

The standard explicit scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta X^2}.$$

Since this scheme is explicit, it is very easy to compute at each time step $n+1$ the value of the approximation $(u_i^{n+1})_i$ from the value of the approximation at the time step n, namely $(u_i^n)_i$.

$$u_i^{n+1} = u_i^n + \Delta t\{\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta X^2}\}$$

Note that here, we may define the scheme S by setting:

$$S(\Delta t, \Delta x, (n+1)\Delta T, i\Delta x, u_i^{n+1}, [u_{i-1}^n, u_i^n, u_{i+1}^n]) = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta X^2}.$$

Next, let us discuss the properties of this scheme: clearly, it is **consistent** with the equation since formally, the truncation error is of order two in space and order one in time. Let us recall how one can calculate the truncation error for a smooth function u with bounded partial derivatives. Simply write the taylor expansions

$$u_{i+1}^n = u_i^n + u_x(n\Delta t, x_i)\Delta X + \frac{1}{2}u_{xx}(n\Delta t, x_i)\Delta X^2 + u_{xxx}\frac{1}{6}\Delta X^3 + \frac{1}{24}u_{xxxx}\Delta X^4 + \Delta X^4\epsilon(\Delta X)$$

and

$$u_{i-1}^n = u_i^n - u_x(n\Delta t, x_i)\Delta X + \frac{1}{2}u_{xx}(n\Delta t, x_i)\Delta X^2 - \frac{1}{6}u_{xxx}\Delta X^3 + \frac{1}{24}u_{xxxx}\Delta X^4 + \Delta X^4\epsilon(\Delta X)$$

Then, adding up the two expansions, substracting $2u_i^n$ from the left- and right hand sides and dividing by ΔX^2, one obtains

$$\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta X^2} = u_{xx} + \frac{1}{12}u_{xxxx}\Delta X^2 + o(\Delta X^2)$$

and thus the truncation error for this approximation of the second spatial derivative is of order 2. Similarly the expansion

$$u_i^{n+1} = u_i^n + u_t(n\Delta t, x_i)\Delta t + \frac{1}{2}u_{tt}(n\Delta t, x_i)\Delta t^2 + \Delta t^2\epsilon(\Delta t)$$

yields

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = u_t(n\Delta t, x_i) + \frac{1}{2}u_{tt}\Delta t + \Delta t\epsilon(\Delta t).$$

The truncation error for the approximation of the first derivative in time is of order 1 only (for more details about computation of truncations errors, see the book by Thomas [7]).

Furthermore, the approximation S is **monotone** if and only if S is decreasing in u_i^n, u_{i+1}^n and u_{i-1}^n. First of all, it is unconditionally decreasing with respect to both u_{i-1}^n and u_{i+1}^n. Secondly, it is only decreasing in u_i^n if the following CFL condition is satisfied:

$$(-1 + 2\frac{\Delta t}{\Delta X^2}) \leq 0$$

or equivalently

$$\Delta t \leq \frac{1}{2}\Delta X^2.$$

The standard implicit scheme

For many financial applications, the explicit scheme turns out to be very inaccurate because the CFL condition forces the time step to be so small that the rounding error dominates the total computational error (computational error=rounding error+truncation error). Most of the time, an implicit scheme is preferred because it is unconditionally convergent, regardless of the size of the time step. We now evaluate the second derivative at time $(n+1)\Delta t$ instead of time $n\Delta t$,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\Delta X^2}.$$

Implementing an algorithm allowing to compute the approximation is less obvious here. This discrete equation may be converted into a linear system of equations and the algorithm will then consist in inverting a tridiagonal matrix. The truncation errors for smooth functions are the same as for the explicit scheme and the consistency follows from this analysis.

We claim that for any choice of the time step, the implicit scheme is **monotone**. In order to verify that claim, let us rewrite the implicit scheme using the notation S:

$$S(\Delta t, \Delta x, (n+1)\Delta T, i\Delta x, u_i^{n+1}, [u_{i-1}^{n+1}, u_i^n, u_{i+1}^{n+1}]) =$$
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\Delta X^2}.$$

Since S is decreasing in u_i^n, u_{i+1}^{n+1} and u_{i-1}^{n+1} the implicit scheme is unconditionally **monotone**.

3.2 The Black-Scholes-Merton PDE

The price of a European call $u(t, x)$ satisfies the degenerate linear PDE

$$u_t + ru - \frac{1}{2}\sigma^2 x^2 u_{xx} - rxu_x = 0 \text{ in } (0, T] \times [0, x)$$
$$u(0, x) = (x - K)^+.$$

The Black-Scholes-Merton PDE is linear and its Elliptic operator is degenerate. The first derivative u_x can be easily approximated in a monotone way using a forward Finite Difference

$$-rxu_x \approx -rx_i \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x}.$$

4 A nonlinear example: The Passport Option

It is an interesting example of a one-dimensional nonlinear HJB equation. I do not present the underlying model here and refer to the article [8] for more details and references. I introduce directly the reduced equation

$$u_t + \gamma u - \sup_{|q| \leq 1} \{((r - \gamma - r_c)q - (r - \gamma - r_t)x)u_x + \frac{1}{2}\sigma^2 (x - q)^2 u_{xx}\}$$
$$u(0, x) = \max(x, 0)$$

where t is the time variable and x is a real number representing the wealth in the trading account per unit of underlying stock. In this example, the solution is no longer bounded but grows at most linearly at infinity. The Barles-Souganidis [3] framework can be slightly modified to accommodate the linear growth of the value function at infinity.

When the payoff is convex, it is easy to see that the optimal value for q is either $+1$ or -1. When the payoff is no longer convex, the supremum may be achieved inside the interval at $q^* = x - \frac{(r-\gamma-r_c)u_x}{\sigma^2 u_{xx}}$. For simplicity, we consider only the convex case.

To simplify further , we focus on a simple case: we assume that $r - \gamma - r_t = 0$ and $r - \gamma - r_c < 0$. This equation is still fairly difficult to solve because the approximation scheme depends on the control q:

$$u_t + \gamma u - \max\{(r - \gamma - r_c)u_x + \frac{1}{2}\sigma^2(x-1)^2 u_{xx},$$
$$-(r - \gamma - r_c)u_x + \frac{1}{2}\sigma^2(x+1)^2 u_{xx}\}$$
$$u(0,x) = \max(x,0)$$

One can easily construct an explicit monotone scheme by using the appropriate forward or backward finite difference for the first partial derivative. Often, this type of scheme is called "upwind" because you move along the direction prescribed by the deterministic dynamics $b(x, \alpha^*)$ corresponding to the optimal control α^* and pick the corresponding neighbor. For instance, for the passport option, the dynamics are

$$\text{For } q^* = 1, b^{\alpha^*}(t,x) = q^*(r - \gamma - r_c) = (r - \gamma - r_c) < 0$$
$$\text{For } q^* = -1, b^{\alpha^*}(t,x) = -(r - \gamma - r_c) > 0$$

and the corresponding upwind Finite Differences are

$$\text{For } q^* = 1, u_x \approx D^- u_i^n$$
$$\text{For } q^* = -1, u_x \approx D^+ u_i^n$$

where we used the standard notations

$$D^- u_i^n = \frac{u_i^n - u_{i-1}^n}{\Delta x}, D^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{\Delta x}.$$

Then the scheme reads

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \gamma u_i^n - \max\{$$
$$(r - \gamma - r_c)\frac{u_i^n - u_{i-1}^n}{\Delta x} + \frac{1}{2}\sigma^2(x_i - 1)^2 \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta x^2},$$
$$-(r - \gamma - r_c)\frac{u_{i+1}^n - u_i^n}{\Delta x} + \frac{1}{2}\sigma^2(x_i + 1)^2 \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta x^2}\} = 0.$$

This scheme clearly satisfies the monotonicity assumption under the CFL condition

$$\Delta t \leq \frac{1}{\gamma + \frac{|r-\gamma-r_c|}{\Delta x} + \frac{\sigma^2 \max\{\max_i\{i\Delta x - 1\}^2, \max_i\{i\Delta x + 1\}^2\}}{\Delta x^2}}.$$

Approximating the first spatial derivative by the classic centered finite difference, i.e. $u_x \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$ would not yield a monotone scheme here.

Note that this condition is very restrictive. First of all, as expected, Δt has to be of order Δx^2. Furthermore, Δt also depends on the size of the grid through the terms $(i\Delta x - 1)^2, (i\Delta x - 1)^2$ and even approaches 0 as the size of the domain goes to infinity. In this situation, we renounce using the above explicit scheme and replace it by the fully implicit upwind scheme which is unconditionally monotone.

$$
\begin{aligned}
&\frac{u_i^{n+1} - u_i^n}{\Delta t} + \gamma u_i^{n+1} - \max\{ \\
&(r - \gamma - r_c)\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + \frac{1}{2}\sigma^2 (x_i - 1)^2 \frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\Delta x^2}, \\
&-(r - \gamma - r_c)\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} + \frac{1}{2}\sigma^2 (x_i + 1)^2 \frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\Delta x^2}\} = 0.
\end{aligned}
$$

Inverting the above scheme is challenging because it depends on the control. This can be done using the classic iterative Howard algorithm which we describe below in a general setting. However, it may be time-consuming to compute the solution of a nonlinear Finite Difference scheme, i.e invert an implicit scheme using an iterative method.

4.1 Howard algorithm

We denote by u_h^n, u_h^{n+1} the approximations at time n and $n+1$. We can rewrite the scheme that we need to invert as

$$\min_{\alpha}\{A_h^\alpha u_h^{n+1} - B_h^\alpha u_h^n\} = 0.$$

Step 0: start with an initial value for the control α_0. Compute the solution v_h^0 of $A_h^{\alpha_0} w - B_h^{\alpha_0} u_h^n = 0$.
Step $k \to k+1$: given v_h^k, find α_{k+1} minimizing $A_h^\alpha v_h^k - B_h^\alpha u_h^n$. Then compute the solution v_h^{k+1} of $A_h^{\alpha_{k+1}} w - B_h^{\alpha_{k+1}} u_h^n = 0$.
Final step: $if |v_h^{k+1} - v_h^k| < \epsilon$, then set $u_h^{n+1} = v_h^{k+1}$.

5 The Bonnans-Zidani [4] approximation

Sometimes, for a given problem, it is very difficult or even impossible to find a monotone scheme. Rewriting the PDE in terms of directional derivatives instead of partial derivatives can be extremely useful. For example, in two spatial dimensions, a naive discretization of the partial derivative v_{xy} may fail to be monotone. In fact, approximating second-order operators with crossed derivatives in a monotone way is not easy. You actually need to be able to interpret you second-order term as a directional derivative (of a linear combination of directional derivatives) and approximate each directional derivative by the adequate Finite Difference. In other words, you need to "move in the right direction" in order to preserve the Elliptic structure of the operator.

Here is for instance a *naive approximation* of v_{xy} (assume $\Delta x = \Delta y$):

$$v_{xy} \approx \frac{v_{i+1,j+1} + v_{i-1,j-1} - v_{i+1,j-1} - v_{i-1,j+1}}{4\Delta x^2}.$$

It is consistent but clearly not monotone (the terms $v_{i-1,j+1}, v_{i+1,j-1}$ have the wrong sign).

Instead, let us look at the second-order derivative:

$$L^\alpha \Phi(t,x) = tr(a^\alpha(t,x) D^2\Phi(t,x))$$

and assume that the coefficients a^α admit the decomposition

$$a^\alpha(t,x) = \sum_\beta \overline{a}^\alpha_\beta \beta\beta^T.$$

The operator can then be expressed in terms of the directional derivatives $D^2_\beta = tr[\beta\beta^T D^2]$

$$L^\alpha \Phi(t,x) = \sum_\beta \overline{a}^\alpha_\beta(t,x) D^2_\beta \Phi(t,x).$$

Finally, we can use the consistent and monotone Bonnans-Zidani [4] approximation for each directional derivative

$$D^2_\beta v(t,x) \approx \frac{v(t, x+\beta\Delta x) + v(t, x-\beta\Delta x) - 2v(t,x)}{|\beta|^2 \Delta x^2}.$$

In practice, if the points $x+\beta\Delta x$, $x-\beta\Delta x$ are not on the grid, you need to estimate the value of v at these points by simple linear interpolation between 2 grid points. Of course, you have to make sure that the interpolation procedure preserves the monotonicity of the approximation.

Comments:

- In all the above examples, I only consider the immediate neighbors of a given point $((n+1)\Delta t, i\Delta x)$, namely $(n\Delta t, i\Delta x)$, $(n\Delta t, (i-1)\Delta x)$, $(n\Delta t, (i+1)\Delta x)$, $((n+1)\Delta t, (i-1)\Delta x$ and $(n+1)\Delta t, (i+1)\Delta x)$. Sometimes, it is worth considering a larger neighborhood and picking neighbors located further away from $((n+1)\Delta t, i\Delta x)$. It is particularly useful for the discretization of a transport term with a high speed, when information "travels fast".

- The theoretical accuracy of a monotone finite difference scheme is quite low. The Barles-Jakobsen theory [2] predicts a typical rate of 1/5 ($|h|^{\frac{1}{5}}$ where $h = \sqrt{\Delta x^2 + \Delta t}$ and an optimal rate of 1/2. Sometimes, higher rates are reported in practice (first order).

6 Working in a finite domain

When one implements a numerical scheme, one cannot work on the whole space and must instead work on a finite grid. Consequently, one has to impose some extra boundary conditions at the edges of the grid. This creates an additional source of error and even sometimes instabilities. Indeed, when the behavior at infinity is not known, imposing an overestimated boundary condition may cause the computed solution to blow up. If the behavior of the solution at infinity is known, it is then relatively easy to come up with a reasonable boundary condition. Next, one can try to prove that the extra error introduced is confined within a boundary layer or more precisely decreases exponentially as a function of the distance to the boundary (see [1] for a result in this direction). Also, one can perform experiments to ensure that theses artificial boundary conditions do not affect the accuracy of the results, by increasing the size of the domain and checking that the first 6 significant digits of the computed solution are not affected.

7 Variational Inequalities and splitting methods

7.1 The American option

This is the easiest example of Variational Inequalities arising in Finance and it gives the opportunity to introduce splitting methods. We look at the simplified VI: $u(t,x)$ solves

$$\max(u_t - u_{xx}, u - \psi(t,x)) = 0 \text{ in } (0,T] \times I\!R \quad (7)$$
$$u(0,x) = u_0(x). \quad (8)$$

This PDE can be approximated using the following semi-discretized scheme
1st Step: Given u^n, solve the heat equation

$$w_t - w_{xx} = 0 \text{ in } (n\Delta t, (n+1)\Delta t] \times I\!R \quad (9)$$
$$w(n\Delta t, x) = u^n(x). \quad (10)$$

and set

$$u^{n+\frac{1}{2}}(x) = w((n+1)\Delta t, x)$$

Step 2

$$u^{n+1}(x) = \inf(u^{n+\frac{1}{2}}(x), \psi((n+1)\Delta t, x))$$

It is quite simple to prove the convergence of a splitting method using the Barles-Souganidis framework. There are many VI arising in Quantitative Finance, in particular in presence of singular controls and splitting methods are extremely useful for this type of HJB equations. We refer to the guest lecture by H. M. Soner for an introduction to singular control and its applications.

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