Interest Rates & Credit Risk — Solutions #1

2. Payment at t_i is $\quad C_i = \dfrac{1}{\Delta t}\left[\dfrac{1}{P_{t_{i-1}}(t_i)} - 1\right]$

value at $t=0$ $\quad V_i = \dfrac{1}{\Delta t}\left(P_0(t_{i-1}) - P_0(t_i)\right)$

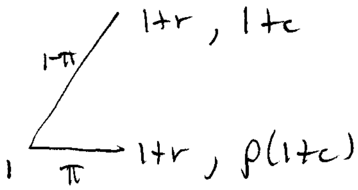
Value of Contract $= \sum_{i=1}^{N} V_i = \dfrac{1}{\Delta t}\left(1 - P_0(t_N)\right)$

3. Floating leg has value $\quad \cdot(1 - P_0(t_N))$

Fixed leg has value $= \sum_{i=1}^{N} K\Delta t\, P_0(t_i)$.

Swap rate $\quad K = \dfrac{1 - P_0(t_N)}{\Delta t \cdot \sum_{i=1}^{N} P_0(t_i)}$

4. (1.)



$$V_0 = 1 = \frac{1}{1+r}\left[(1-\pi)(1+c) + p\pi(1+c)\right]$$

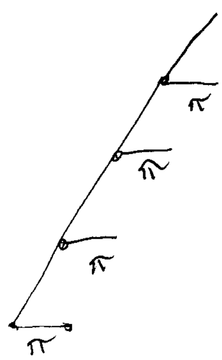
$$\Rightarrow \quad c = \frac{1+r}{1-\pi+p\pi} - 1$$

(2.) $\quad \log(1+c) - \log(1+r) = -\log(1-\pi+p\pi)$

$$\sim \pi(1-p) + O(\pi^2(1-p)^2)$$

$\log(1+x) = x + O(x^2)$ if $x \ll 1$

(3.)



Value of $T=2$ bond in non default state at $T=1$ $= \$1$

in default state $= \$p$

Option value at $T=1$ $= \begin{cases} 1-\frac{1}{2} & \text{non. def} \\ (p-\frac{1}{2})^+ & \text{def.} \end{cases}$

Option Value at $T=0 = \dfrac{1}{1+r}\left[(1-\pi)\left(\tfrac{1}{2}\right) + \pi\left(p - \tfrac{1}{2}\right)^+\right]$

4.4 Value at $t=3$ if no default (SOME VARIATIONS ARE POSSIBLE!)

$$V_3 = \frac{1}{1+r}\Big[\underbrace{S}_{\text{premium}} - \underbrace{\pi(1-\rho)(1+c)}_{\text{default payment}}\Big]$$

At $t=2$ if no default

$$V_2 = \frac{1}{1+r}\Big[\underbrace{S}_{\text{premium}} + \underbrace{(1-\pi)V_3}_{\text{value of remaining CDS}} - \underbrace{\pi(1-\rho)(1+c)}_{\text{def. payment}}\Big] = V_3 + \left(\frac{1-\pi}{1+r}\right)V_3$$

At $t=1$

$$V_1 = \frac{1}{1+r}\Big[S + (1-\pi)V_2 - \pi(1-\rho)(1+c)\Big] = V_3 + \left(\frac{1-\pi}{1+r}\right)V_2$$

At $t=0$:

$$V_0 = \frac{1}{1+r}\Big[S + (1-\pi)V_1 - \pi(1-\rho)(1+c)\Big] = V_3 + \left(\frac{1-\pi}{1+r}\right)V_1$$

$$= V_3\left[1 + \left(\frac{1-\pi}{1+r}\right) + \left(\frac{1-\pi}{1+r}\right)^2 + \left(\frac{1-\pi}{1+r}\right)^3\right]$$

NOTE: Swap Rate $S^* = \pi(1-\rho)(1+c)$

5. From (3.24), (3.25) with $\kappa = -k$, $\eta = k\theta$, $\delta = \sigma^2$, $\gamma = 0$

$$\begin{cases} \partial_t A = -k\theta B - \frac{1}{2}\sigma^2 B^2 & A(T) = 0 \\ \partial_t B = kB + 1 & B(T) = 0 \end{cases}$$

B-integration: $-\dfrac{2}{\sigma^2}\displaystyle\int_{B(t,T)}^{B(T,T)} \frac{dB}{B(B+2k\theta/\sigma^2)} = \int_t^T ds$

$$\text{LHS} = -\frac{2}{\sigma^2}\cdot\frac{\sigma^2}{2k\theta}\Big[\log|B(s,T)| - \log\Big|B(s,T)+\frac{2k\theta}{\sigma^2}\Big|\Big]_t^T$$

$$= \frac{1}{k\theta}\left[\log\left|\frac{B(s,T)}{B(s,T)+2k\theta/\sigma^2}\right|\right] = (T-t) = \text{RHS}$$

B-integration
$$\int_{B(t,T)}^{B(T,T)} \frac{dB}{k(B+1/k)} = \int_t^T ds$$

$$LHS = \frac{1}{k}\log\left|\frac{B(T,T)+\frac{1}{k}}{B(t,T)+\frac{1}{k}}\right| = T-t = RHS$$

$$kB(t,T)+1 = e^{-k(T-t)}$$

$$B(t,T) = \frac{1}{k}\left[e^{-k(T-t)}-1\right]$$

A-integration
$$A(T,T)-A(t,T) = \int_{B(t,T)}^{B(T,T)} \left(\frac{\partial_t A}{\partial_t B}\right).dB$$

$$= -\frac{\sigma^2}{2}\int_{B(t,T)}^{B(T,T)} \frac{\left(B+\frac{2k\Theta}{\sigma^2}\right)B}{kB+1}\, dB$$

Long division :
$$\frac{B^2+\frac{2k\Theta B}{\sigma^2}}{kB+1} = \frac{1}{k}B + \frac{1}{k^2}\left(\frac{2k\Theta}{\sigma^2}-1\right) - \frac{1}{k^2}\left(\frac{2k^2\Theta}{\sigma^2}-1\right)\left(\frac{1}{kB+1}\right)$$

use B-integral

$$\therefore A(t,T) = \frac{\sigma^2}{2}\left[-\frac{1}{2k}B^2(t,T) - \frac{1}{k^2}\left(\frac{2k^2\Theta}{\sigma^2}-1\right)B(t,T) - \frac{1}{k^2}\left(\frac{2k^2\Theta}{\sigma^2}-1\right)(T-t)\right.$$

6. 1. $f_t(T) = -\frac{\partial}{\partial T} \log P_t(T)$ ④

$= -\frac{\partial}{\partial T}\left[A(t,T) + B(t,T) r_t\right]$

$$\boxed{f(t,T) = \frac{\partial}{\partial t}\left[A(t,T) + B(t,T) r_t\right]}$$

2. $f(t,T) = -\eta(t) B(t,T) - \frac{1}{2}\delta(t) B^2(t,T)$

$+ \left(1 - \kappa(t) B(t,T) - \frac{1}{2}\gamma(t) B(t,T)^2\right) r_t := C(t) + D(t) r_t$

$\lim_{T \downarrow t} f(t,T) = -\eta(t) B(t,t) - \frac{1}{2}\delta(t) B(t,t)$

$+ \left(1 - \kappa(t) B(t,t) - \frac{1}{2}\gamma(t) B(t,t)^2\right) r_t$

$= r_t$

3. In Vasicek $r_t \sim N(\mu(t), v^2(t))$

$\mu(t) = e^{-kt} r_0 + (1 - e^{-kt})\theta$

$v^2(t) = \frac{\sigma^2}{2k}(1 - e^{-2kt})$

$\therefore f_t(T) \sim N(\mu_f(t), v_f^2(t))$

Where $\mu_f(t) = C(t) + D(t)\mu(t)$ <u>C, D as above.</u>

$v_f^2(t) = (D(t))^2 v^2(t)$

$C(t) = -k\theta B(t,T) - \frac{1}{2}\sigma^2 B(t,T)$

$D(t) = 1 + kB.$

7. $dZ_t = \left[k(\theta_x - X_t) + k(\theta_y - Y_t)\right] dt + \sigma\left(\sqrt{X_t}\, dW_t^1 + \sqrt{Y_t}\, dW_t^2\right)$

$= k\left[(\theta_x + \theta_y) - Z_t\right] dt + \sigma\sqrt{Z_t}\, d\widetilde{W}_t$

where $d\widetilde{W}_t = \frac{1}{\sqrt{Z_t}}\left(\sqrt{X_t}\, dW_t^1 + \sqrt{Y_t}\, dW_t^2\right)$

since $\widetilde{W}$ is cts, $\widetilde{W}(0) = 0$, a mg and $[\widetilde{W}, \widetilde{W}]_t = t$

$\widetilde{W}$ is a <u>BM</u> (by Levy).

8. Vasicek $\begin{cases} \partial_t B = kB + c_1 \\ B(T,T) = -c_2 \end{cases}$ (5)

Then $$\int_{B(t,T)}^{B(T,T)} \frac{dB}{k\left[B + c_1/k\right]} = T-t.$$

$$\therefore \frac{1}{k}\log\left|\frac{-c_2 + c_1/k}{B(t,T)+c_1/k}\right| = T-t$$

$$B(t,T) = -c_1/k + \left(\frac{c_1}{k} - c_2\right)e^{-k(T-t)}$$

9. ANS Swap rate = 0.0161