Convergence results for the indifference value based on the stability of BSDEs

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Brownian setting with variable correlation

Convergence problem?
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Characterization

Brownian setting with variable correlation

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Convergence results for the indifference value
1. Indifference valuation
Financial market:

- Risk-free bank account yielding zero interest
- Risky asset with price process $S = (S_t)_{0 \leq t \leq T}$
- Financial product with payoff $H$ at time $T$
- In mathematical terms, $S$ is a semimartingale and $H$ a random variable on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. 

Investor's utility if (s)he has capital $x \in \mathbb{R}$. 

Assumption: The investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. 

In mathematical terms, the investor's utility is $U(x) = \hat{U}(x) = -\exp(-\gamma x)$. 

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Financial market:
- Risk-free bank account yielding zero interest
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Problem formulation:
- Valuation of $H$ based on the risk preferences of an investor
- Assumption: The investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$
- $U(x) \triangleq$ Investor’s utility if (s)he has capital $x \in \mathbb{R}$. 
The **indifference value** $h$ of $H$ is implicitly defined by

$$
\sup_{\vartheta \in \mathcal{A}} E \left[ U \left( \int_0^T \vartheta_t \, dS_t \right) \right] = \sup_{\vartheta \in \mathcal{A}} E \left[ U \left( \int_0^T \vartheta_t \, dS_t + H - h \right) \right],
$$

where $\mathcal{A}$ is the set of admissible trading strategies.
Definition

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where $\mathcal{A}$ is the set of admissible trading strategies.

The value $h$ makes the investor indifferent (in terms of maximal expected utility) between buying $H$ for the amount $h$ and not buying $H$. 
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source: www.myownproperty.co.uk
\[ U(x) = -\exp(-\gamma x) \text{ for a fixed } \gamma > 0 \]

\[ \Downarrow \text{ direct calculation} \]

The indifference value \( h \) is given by

\[ h = \frac{1}{\gamma} \log \frac{V^0}{V^H}, \]

\[ V^H := \inf_{\vartheta \in \mathcal{A}} E \left[ \exp \left( -\int_0^T \gamma \vartheta_t \, dS_t - \gamma H \right) \right]. \]
$U(x) = -\exp(-\gamma x)$ for a fixed $\gamma > 0$

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The indifference value $h$ is given by

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$\downarrow$

The focus lies on $V^H$. 

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The underlying model:

- Two Brownian motions $W$ and $Y$ have constant instantaneous correlation $\rho$; i.e., $W = \rho Y + \sqrt{1 - \rho^2} Y^\perp$ for a Brownian motion $Y^\perp$ independent from $Y$. 

Example: Executive stock options
Manager receives options $H$.
Because of legal restrictions, (s)he can hedge $H$ only partially by trading in a correlated stock or an index.
The underlying model:

- Two Brownian motions $W$ and $Y$ have constant instantaneous correlation $\rho$; i.e., $W = \rho Y + \sqrt{1 - \rho^2} Y_\perp$ for a Brownian motion $Y_\perp$ independent from $Y$.
- The traded stock $S$ is given by
  \[
  \frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t, \quad 0 \leq t \leq T, \quad S_0 > 0.
  \]
- Assumption: $\mu$ and $\sigma$ are predictable with respect to $(Y_t)_{0 \leq t \leq T}$, the filtration generated by $Y$.
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Example: Executive stock options
- Manager receives options $H$.
- Because of legal restrictions, (s)he can hedge $H$ only partially by trading in a correlated stock or an index.
Proposition (An explicit formula; Tehranchi 2004)

Under boundedness assumptions, one has

\[ V^H = \left( E_{\widehat{P}} \left[ \exp \left( -\gamma H - \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} \, dt \right)^{1-\rho^2} \right] \right)^{\frac{1}{1-\rho^2}}, \]

where the probability measure \( \widehat{P} \) is given by

\[ \frac{d\widehat{P}}{dP} := \exp \left( - \int_0^T \frac{\mu_t}{\sigma_t} \, dW_t - \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} \, dt \right). \]
Variable correlation:

So far: \( W_t = \rho Y_t + \sqrt{1 - \rho^2} Y_t^\perp \)
\[ = \int_0^t \rho \, dY_s + \int_0^t \sqrt{1 - \rho^2} \, dY_s^\perp \]
with constant \( \rho \)
Variable correlation:

- So far: $W_t = \rho Y_t + \sqrt{1 - \rho^2} Y_t^\perp$
  \hspace{1cm} $= \int_0^t \rho \, dY_s + \int_0^t \sqrt{1 - \rho^2} \, dY_s^\perp$ \hspace{0.5cm} with constant $\rho$

- Now: $W_t = \int_0^t \rho_s \, dY_s + \int_0^t \sqrt{1 - \rho_s^2} \, dY_s^\perp$ \hspace{0.5cm} with variable $\rho$
Variable correlation:

- So far: \( W_t = \rho Y_t + \sqrt{1 - \rho^2} Y_t^\perp \)
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- Now: \( W_t = \int_0^t \rho_s \, dY_s + \int_0^t \sqrt{1 - \rho_s^2} \, dY_s^\perp \text{ with variable } \rho \)

Proposition (Bounds; Frei and Schweizer 2008)

For \( (\mathcal{Y}_t)_{0 \leq t \leq T} \)-predictable \( \rho \) with boundedness assumptions,

\[
\left( E_\hat{\mathbb{P}} \left[ \exp \left( \hat{H}^{1/\delta} \right) \right] \right)^\delta \leq V^H \leq \left( E_\hat{\mathbb{P}} \left[ \exp \left( \hat{H}^{1/\delta} \right) \right] \right)^\delta,
\]

where \( \hat{H} := -\gamma H - \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} \, dt \) and

\[
\bar{\delta} := \sup_{t \in [0, T]} \left\| \frac{1}{1 - \rho_t^2} \right\|_{L^\infty}, \quad \delta := \inf_{t \in [0, T]} \frac{1}{\left\| 1 - \rho_t^2 \right\|_{L^\infty}}.
\]
Ideas for an approximation of $V^H$:

1. If $\rho$ is piecewise constant in time, there is an explicit formula for $V^H$. 

\begin{figure}
\centering
\begin{tikzpicture}
\begin{axis}[
    xlabel={time},
    ylabel={piecewise constant process},
    xmin=-1, xmax=1,
    ymin=-1, ymax=1,
    xtick={-1,-0.5,0,0.5,1},
    ytick={-1,-0.5,0,0.5,1},
    grid=both,
    ]
\addplot[red, mark=*] coordinates {
    (-1,1)
    (-0.5,0.5)
    (0,0)
    (0.5,-0.5)
    (1,-1)
};
\end{axis}
\end{tikzpicture}
\end{figure}
Ideas for an approximation of $V^H$:

1. If $\rho$ is piecewise constant in time, there is an explicit formula for $V^H$.

2. Approximate a general $\rho$ by a sequence $(q_n)_{n \in \mathbb{N}}$ of piecewise constant processes.
Ideas for an approximation of $V^H$:

1. If $\rho$ is piecewise constant in time, there is an explicit formula for $V^H$.

2. Approximate a general $\rho$ by a sequence $(q_n)_{n \in \mathbb{N}}$ of piecewise constant processes.

3. Show that values corresponding to $q_n$ converge to $V^H$.

Problem: It is difficult to show this directly. 

→ study BSDE
2. A convergence result for BSDEs
Let $B$ be a $d$-dimensional Brownian motion and consider

$$d\Gamma_t = f(t, Z_t) \, dt + Z_t \, dB_t, \quad 0 \leq t \leq T, \quad \Gamma_T = H,$$

where

- $f : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$
- $H$ is a bounded random variable.
Let $B$ be a $d$-dimensional Brownian motion and consider
\[ d\Gamma_t = f(t, Z_t) \, dt + Z_t \, dB_t, \quad 0 \leq t \leq T, \quad \Gamma_T = H, \]
where
\begin{itemize}
  \item $f : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$
  \item $H$ is a bounded random variable.
\end{itemize}

The results hold not only in a Brownian setting, but more generally in a continuous filtration (i.e., a filtration where any local martingale has a continuous version).
Theorem (Convergence of BSDEs)

Fix $t \in [0, T]$ and let $(f^n, H^n)_{n=1,2,...,\infty}$ be a sequence of parameters such that

- $f^n$ satisfy some quadratic-growth and local-Lipschitz conditions in $z$ (uniformly in $n = 1, \ldots, \infty$);
- $\lim_{n \to \infty} H^n = H^\infty$ a.s. and for almost all $(s, \omega) \in [t, T] \times \Omega$, $\lim_{n \to \infty} f^n(s, z)(\omega) = f^\infty(s, z)(\omega)$ for all $z \in \mathbb{R}^d$.

Then there exist unique solutions $(\Gamma^n, Z^n)$ with parameters $(f^n, H^n)$ for $n = 1, \ldots, \infty$, and

$$
\lim_{n \to \infty} \Gamma^n_t = \Gamma^\infty_t \text{ a.s., } \lim_{n \to \infty} E \left[ \int_t^T |Z^n_s - Z^\infty_s|^2 \, ds \right] = 0.
$$
Corollary (Special form of $f^n$)

Suppose additionally that

- $H^n$ converges to $H^\infty$ in $L^\infty$ as $n \to \infty$;
- there exist sequences $(d^n)_{n \in \mathbb{N}}$ and $(\bar{d}^n)_{n \in \mathbb{N}}$ of deterministic functions which converge to 1 uniformly on $[t, T]$ such that $f^n = d^n f + \bar{d}^n f$ for every $n = 1, \ldots, \infty$.

Then we have

$$\sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \to 0 \quad \text{in } L^\infty \quad \text{as } n \to \infty.$$
3. Applying the convergence result
A BSDE characterization of $V^H$

Revisiting the nontradable asset model:

- Two Brownian motions $W$ and $Y$ have time-dependent instantaneous correlation $\rho$; $dW_t = \rho_t \, dY_t + \sqrt{1 - \rho_t^2} \, dY^\perp_t$ for a Brownian motion $Y^\perp$ independent from $Y$.

- The traded stock $S$ is given by
  \[ \frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t, \quad 0 \leq t \leq T, \quad S_0 > 0. \]

- Assumptions: $\mu$ and $\sigma$ are predictable with respect to $(\mathcal{Y}_t)_{0 \leq t \leq T}$, the filtration generated by $Y$. The nontradable claim $H$ is $\mathcal{Y}_T$-measurable.

- The indifference value $h$ is given by $h = \frac{1}{\gamma} \log \frac{V^0}{V^H}$, where
  \[ V^H := \inf_{\vartheta \in A} E \left[ \exp \left( - \int_0^T \gamma \vartheta_t \, dS_t - \gamma H \right) \right]. \]
A BSDE characterization of $V^H$:

We have $V^H = \exp(-\gamma \Gamma_0)$, where $\Gamma$ solves the BSDE

$$
\text{d}\Gamma_t = \left( \frac{\gamma}{2} (1 - \rho_t^2) Z_t^2 + \rho_t \lambda_t Z_t - \frac{\lambda_t^2}{2\gamma} \right) \text{d}t + Z_t \text{d}Y_t, \quad \Gamma_T = H
$$

with $\lambda := \mu/\sigma$. 
A BSDE characterization of $V^H$:
We have $V^H = \exp(-\gamma \Gamma_0)$, where $\Gamma$ solves the BSDE

$$d\Gamma_t = \left(\frac{\gamma}{2} (1 - \rho_t^2) Z_t^2 + \rho_t \lambda_t Z_t - \frac{\lambda_t^2}{2\gamma}\right)dt + Z_t dY_t, \quad \Gamma_T = H$$

with $\lambda := \mu/\sigma$.

In the notation of the second part:

$$d\Gamma_t = f(t, Z_t) dt + Z_t dB_t, \quad \Gamma_T = H,$$

where $B := Y$ and $f(t, z) := \frac{\gamma}{2} (1 - \rho_t^2) z^2 + \rho_t \lambda_t z - \frac{\lambda_t^2}{2\gamma}$.
A BSDE characterization of $V^H$:
We have $V^H = \exp(-\gamma \Gamma_0)$, where $\Gamma$ solves the BSDE
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\]
with $\lambda := \mu / \sigma$.

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d\Gamma_t = f(t, Z_t) dt + Z_t dB_t, \quad \Gamma_T = H,
\]
where $B := Y$ and $f(t, z) := \frac{\gamma}{2} (1 - \rho_t^2) z^2 + \rho_t \lambda_t z - \frac{\lambda_t^2}{2\gamma}$

Remark:
The application can be done for $(Y_t)_{0 \leq t \leq T}$-predictable $\rho$, but we consider here only a deterministic, time-dependent $\rho$. 
If $\rho$ is piecewise constant in time, there is an explicit formula for the solution of the BSDE.
An approximation of $V^H$

1. If $\rho$ is piecewise constant in time, there is an explicit formula for the solution of the BSDE.

2. Approximate a general $\rho$ by a sequence $(q_n)_{n \in \mathbb{N}}$ of piecewise constant processes.
An approximation of $\mathcal{V}^H$

1. If $\rho$ is piecewise constant in time, there is an explicit formula for the solution of the BSDE.

2. Approximate a general $\rho$ by a sequence $(q_n)_{n \in \mathbb{N}}$ of piecewise constant processes.

3. Apply the convergence result to show the convergence of the solutions of the corresponding BSDEs.
1. Step: Piecewise constant processes

Let \( q : [0, T] \rightarrow ]-1, 1[ \) be of the form

\[
q = q^1 1_{\{t_0\}} + \sum_{j=1}^{n} q^j 1_{[t_{j-1}, t_j]} \quad \text{for} \quad t = t_0 \leq t_1 \leq \cdots \leq t_n = T.
\]
1. Step: Piecewise constant processes

Let $q : [0, T] \to ]-1, 1[$ be of the form

$$q = q^1 \mathbb{1}_{\{t_0\}} + \sum_{j=1}^{n} q^j \mathbb{1}_{[t_{j-1}, t_j]} \quad \text{for} \ t = t_0 \leq t_1 \leq \cdots \leq t_n = T.$$ 

Then the BSDE

$$d\Gamma^q_t = \left( \frac{\gamma}{2} (1 - q^2_t) |Z^q_t|^2 + \rho_t \lambda_t Z^q_t - \frac{\lambda^2_t}{2\gamma} \right) dt + Z^q_t dY_t, \quad \Gamma_T = H$$

has the explicit solution $\Gamma^q_0$ with $\exp(-\gamma \Gamma^q_0)$ equal to

$$E_{\hat{P}} \left[ \cdots E_{\hat{P}} \left[ E_{\hat{P}} \left[ e^{\hat{H}(1-|q^n|^2)} \left| \mathcal{Y}_{t_{n-1}} \right|^{\frac{1-|q^{n-1}|^2}{1-|q^n|^2}} \left| \mathcal{Y}_{t_{n-2}} \right|^{\frac{1-|q^{n-2}|^2}{1-|q^{n-1}|^2}} \cdots \right] \right] \right]^{\frac{1}{1-|q^1|^2}}$$

where

$$\hat{H} := -\gamma H - \frac{1}{2} \int_0^T \lambda_t^2 dt, \quad \frac{d\hat{P}}{dP} := \exp \left( - \int_0^T \lambda_t dW_t - \frac{1}{2} \int_0^T \lambda_t^2 dt \right).$$
2. Step: The approximation of $\rho$

**Question:** Which functions $\rho : [0, T] \rightarrow [-1, 1]$ can be approximated pointwise by piecewise constant functions?
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**Question:** Which functions $\rho : [0, T] \rightarrow [-1, 1]$ can be approximated pointwise by piecewise constant functions?

**Idea:** This approximation is reminiscent of the construction of the Riemann integral.
Recall that a bounded function $g : [0, T] \rightarrow \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[0, T]$. 
Recall that a bounded function $g : [0, T] \to \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[0, T]$.

Assume that $\rho : [0, T] \to [-1, 1]$ is Riemann integrable. Let

$$0 = t_0^n \leq t_1^n \leq \cdots \leq t_{\ell_n}^n = T, \quad s_j^n \in [t_{j-1}^n, t_j^n]$$

be partitions with $\lim_{n \to \infty} (\max_{1 \leq j \leq \ell_n} (t_j^n - t_{j-1}^n)) = 0$ and set $q^n := \sum_{j=1}^{\ell_n} \rho(s_j^n)1_{[t_{j-1}^n, t_j^n]}$. Then

$$\lim_{n \to \infty} q^n(x) = \rho(x) \quad \text{for almost all } x \in [0, T].$$
3. Step: The application of the convergence result

Theorem (Approximating $V^H$)

Assume that $\rho$ is Riemann integrable and $]-1, 1[$-valued. Let

$$0 = t_0^n \leq t_1^n \leq \cdots \leq t_{\ell_n}^n = T, \quad s_j^n \in [t_{j-1}^n, t_j^n]$$

be partitions with $\lim_{n \to \infty} (\max_{1 \leq j \leq \ell_n} (t_j^n - t_{j-1}^n)) = 0$. Then

$$V^H = \lim_{n \to \infty} E_{\hat{\rho}} \left[ \cdots E_{\hat{\rho}} \left[ e^{\hat{H}(1 - |\rho(s_{\ell_n}^n)|^2)} \left| \mathcal{Y}_{t_{\ell_n-1}^n} \right|^{1 - |\rho(s_{n-1}^n)|^2} \right]^{1 - |\rho(s_1^n)|^2} \right]^{1 - |\rho(s_{\ell_n}^n)|^2}$$

with $\hat{H} := -\gamma H - \frac{1}{2} \int_0^T \lambda_t^2 \, dt$. 
Overview

Indifference valuation

BSDE

Characterization

BSDE methods

Approximation of the indifference value

Application

Convergence result for BSDEs

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Convergence results for the indifference value
Thank you very much for your attention!
Admissible strategies

$\mathcal{A}$ consists of all predictable $\vartheta = (\vartheta_t)_{0 \leq t \leq T}$ such that

$$\int_0^T \vartheta_t^2 \, dt < \infty \text{ a.s. and}$$

$$\left( \exp(-\gamma \int_0^t \vartheta_s \, dS_s) \right)_{0 \leq t \leq T}$$

is of class $(D)$.

The latter means that the set

$$\left\{ \exp(-\gamma \int_0^\tau \vartheta_s \, dS_s) \mid \tau \text{ is a stopping time} \right\}$$

is uniformly integrable.
Alternative measurability conditions

Assumptions:

- \( \mu, \sigma \) are predictable w.r.t. the filtration generated by \( W \).
- \( H \) is \( \hat{Y}_T \)-measurable, where \( \hat{Y}_T \) is the sigma-field generated by \( \hat{Y}_t := Y_t + \int_0^t \rho_s \mu_s \sigma_s \, ds, 0 \leq t \leq T \).

Proposition (Bounds; Frei and Schweizer 2008)

For general \( \rho \) with boundedness assumptions, one has

\[
\left( E_{\hat{P}} \left[ \exp \left( \frac{\hat{H}}{\delta} \right) \right] \right)^{\bar{\delta}} \leq V^H \leq \left( E_{\hat{P}} \left[ \exp \left( \frac{\hat{H}}{\delta} \right) \right] \right)^{\delta},
\]

where \( \hat{H} := -\gamma H - \frac{1}{2} E_{\hat{P}} \left[ \int_0^T \frac{\mu_t^2}{\sigma_t^2} \, dt \right] \) and

\[
\bar{\delta} := \sup_{t \in [0, T]} \frac{1}{1 - \rho_t^2} \| L^\infty, \quad \delta := \inf_{t \in [0, T]} \frac{1}{1 - \rho_t^2} \| L^\infty. \]
A general BSDE characterization of $V^H$

Without measurability assumptions on $\rho$, $\mu$, $\sigma$ and $H$:
From Hu, Imkeller and Müller (2005), we have

$$V^H = \exp(-\gamma \Gamma_0),$$

where $\Gamma$ solves the $(\mathcal{F}_t)_{0 \leq t \leq T}$-BSDE

$$d\Gamma_t = \left(\frac{\gamma}{2} \hat{Z}_t^2 - \lambda_t \hat{Z}_t - \frac{\lambda^2_t}{2\gamma}\right) dt + \hat{Z}_t dW_t + \hat{Z}_t dW^\perp_t, \quad \Gamma_T = H$$

for a Brownian motion $W^\perp$ independent of $W$, and $\lambda := \mu/\sigma$.

**Problem:** This BSDE cannot be approximated by a BSDE with an explicit solution.
Approximation under stochastic correlation

Theorem (Approximating $V^H$)

Assume that $\rho$ is $(\mathcal{Y}_t)_{0 \leq t \leq T}$-predictable, left-continuous and $]-1, 1]$-valued. Let $(0 = \tau_0^n \leq \cdots \leq \tau_{\ell_n}^n = T)_{n \in \mathbb{N}}$ be $(\mathcal{Y}_t)_{0 \leq t \leq T}$-stopping times with $\lim_{n \to \infty} \left( \max_{1 \leq j \leq \ell_n} (\tau_j^n - \tau_{j-1}^n) \right) = 0$ a.s. Then we have

$$V^H = \lim_{n \to \infty} E_{\hat{P}} \left[ \cdots E_{\hat{P}} \left[ e^{\hat{H}(1 - |\rho_{\tau_{\ell_n-2}}|^2)} \left| \mathcal{Y}_{\tau_{\ell_n-1}}^n \right|^{1 - |\rho_{\tau_{\ell_n-2}}|^2} \left| \mathcal{Y}_{\tau_{\ell_n-1}}^n \right|^{1 - |\rho_{\tau_{\ell_n-2}}|^2} \right] \right]^{1 - |\rho_{\tau_0^n}|^2}$$

with $\hat{H} := -\gamma H - \frac{1}{2} \int_0^T \lambda_t^2 \, dt$. 

Christoph Frei

Convergence results for the indifference value
Convergence of BSDEs in a continuous filtration

Setting:

- Assume that $\mathbb{F}$ is a general continuous filtration, i.e., all local martingales are continuous.
- Fix an $\mathbb{R}^d$-valued local martingale $M = (M_t)_{0 \leq t \leq T}$.
- Take a nondecreasing and bounded process $D$ such that $\langle M^i \rangle \ll D$ for all $j = 1, \ldots, n$, e.g., $D = \arctan(\sum_{j=1}^n \langle M^i \rangle)$.
Convergence of BSDEs in a continuous filtration

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We consider the BSDE

$$d\Gamma_t = f(t, Z_t) dD_t + \frac{\beta}{2} d\langle N \rangle_t + Z_t dM_t + dN_t, \quad 0 \leq t \leq T, \quad \Gamma_T = H,$$

where

- $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$;
- $\beta \in \mathbb{R}$;
- $H$ is a bounded random variable.
A solution is a triple \((\Gamma, Z, N)\), where

- \(\Gamma\) is a bounded continuous semimartingale;
- \(Z\) is a predictable process with \(E\left[\int_0^T Z_t' d\langle M\rangle_t Z_t\right] < \infty\);
- \(N\) is a square-integrable martingale null at 0 and strongly orthogonal to \(M\).
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d\Gamma_t = f(t, Z_t) \, dD_t + \frac{\beta}{2} \, d\langle N \rangle_t + Z_t \, dM_t + dN_t, \quad 0 \leq t \leq T, \quad \Gamma_T = H,
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A solution is a triple \((\Gamma, Z, N)\), where
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\]
A solution is a triple \((\Gamma, Z, N)\), where

- \(\Gamma\) is a bounded continuous semimartingale;
- \(Z\) is a predictable process with \(E\left[\int_0^T Z'_t \, d\langle M\rangle_t Z_t\right] < \infty\);
- \(N\) is a square-integrable martingale null at 0 and strongly orthogonal to \(M\).

\[
d\Gamma_t = f(t, Z_t) \, dD_t + \frac{\beta}{2} \, d\langle N\rangle_t + Z_t \, dM_t + dN_t, \quad 0 \leq t \leq T, \quad \Gamma_T = H,
\]
Theorem (Convergence of BSDEs)

Fix $t \in [0, T]$ and let $(f^n, \beta^n, H^n)_{n=1,2,\ldots,\infty}$ be a sequence of parameters such that

- $f^n$ satisfy some quadratic-growth and local-Lipschitz conditions in $z$ (uniformly in $n = 1, \ldots, \infty$);
- $\lim_{n \to \infty} \beta^n = \beta^\infty$, $\lim_{n \to \infty} H^n = H^\infty$ a.s. and for $(D \otimes P)$-almost all $(s, \omega) \in [t, T] \times \Omega$,
  $\lim_{n \to \infty} f^n(s, z)(\omega) = f^\infty(s, z)(\omega)$ for all $z \in \mathbb{R}^d$.

Then there exist unique solutions $(\Gamma^n, Z^n, N^n)$ with parameters $(f^n, \beta^n, H^n)$ for $n = 1, \ldots, \infty$, and

$$\lim_{n \to \infty} \Gamma^n_t = \Gamma^\infty_t \text{ a.s.}, \quad \lim_{n \to \infty} E\left[ \langle N^n - N^\infty \rangle_T - \langle N^n - N^\infty \rangle_t \right] = 0,$$

$$\lim_{n \to \infty} E\left[ \int_t^T (Z^n_s - Z^\infty_s)' \, d\langle M \rangle_s (Z^n_s - Z^\infty_s) \right] = 0.$$
Precise assumptions of the convergence result

- There exist a nonnegative predictable $\kappa^1$ with
  \[ \| \int_0^T \kappa^1_s \, ds \|_{L^\infty} < \infty \] and a constant $c^1$ such that
  \[ |f^n(s, z)| \leq \kappa^1_s + c^1|z|^2 \]
  for all $s \in [0, T]$, $z \in \mathbb{R}^d$ and $n = 1, \ldots, \infty$.

- There exist a nonnegative predictable $\kappa^2$ with
  \[ \| \int_0^T |\kappa^2_s|^2 \, ds \|_{L^\infty} < \infty \] and a constant $c^2$ such that
  \[ |f^n(s, z^1) - f^n(s, z^2)| \leq c^2 (\kappa^2_s + |z^1| + |z^2|) |z^1 - z^2| \]
  for all $s \in [0, T]$, $z^1, z^2 \in \mathbb{R}^d$ and $n = 1, \ldots, \infty$. 
References


- Frei: Convergence results for the indifference value based on the stability of BSDEs. Available under http://www.cmap.polytechnique.fr/~frei

