Outperforming The Market Portfolio With A Given Probability

Yu-Jui Huang

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1 Introduction
   - The Problem
   - Related Work
   - The Model

2 On Quantile Hedging

3 The PDE Characterization
   - Stochastic Control Problem Formulation
   - Associated PDE
Consider a financial market with a bond $B(\cdot) = 1$ and $d$ stocks $X = (X_1, \cdots, X_d)$ which satisfy for $i = 1; \cdots d$,

$$dX_i(t) = X_i(t) \left( b_i(X(t))dt + \sum_{k=1}^{d} s_{ik}(X(t))dW_k(t) \right).$$

(1)

Let $\mathcal{H}$ be the set of $\mathbb{F}$-progressively measurable processes $\pi : [0, T) \times \Omega \rightarrow \mathbb{R}^d$, which satisfies

$$\int_0^T \left( |\pi'(t)\mu(X(t))| + \pi'(t)\alpha(X(t))\pi(t) \right) dt < \infty, \quad \text{a.s.},$$

in which $\mu = (\mu_1, \cdots, \mu_d)$ and $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d}$ with

$\mu_i(x) = b_i(x)x_i$, $\sigma_{ik}(x) = s_{ik}(x)x_i$, and $\alpha(x) = \sigma(x)\sigma'(x)$. 

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in which \( \mu = (\mu_1, \cdots, \mu_d) \) and \( \sigma = (\sigma_{ij})_{1 \leq i, j \leq d} \) with \( \mu_i(x) = b_i(x)x_i \), \( \sigma_{ik}(x) = s_{ik}(x)x_i \), and \( \alpha(x) = \sigma(x)\sigma'(x) \).
For each $\pi \in \mathcal{H}$ and initial wealth $y \geq 0$ the associated wealth process will be denoted by $Y^{y,\pi}(\cdot)$. This process solves

$$dY^{y,\pi}(t) = Y^{y,\pi}(t) \sum_{i=1}^{d} \pi_i(t) \frac{dX_i(t)}{X_i(t)}, \quad Y^{y,\pi}(0) = y.$$  

In this paper, we want to determine and characterize

$$V(T, x, p) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\},$$

where $X(0) = x$, $g : (0, \infty)^d \mapsto \mathbb{R}_+$ is a measurable function.
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where $X(0)=x$, $g : (0, \infty)^d \rightarrow \mathbb{R}_+$ is a measurable function.
In the case where $p = 1$ and $g(x) = x_1 + \cdots + x_d$,

$$\tilde{V}(T, x, 1) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t. } Y^{y, \pi}(T) \geq g(X(T)) \text{ a.s.}\}.$$ 

In Fernholz and Karatzas (2008), a PDE characterization for $\tilde{V}(T, x, 1) := V(T, x, 1)/g(x)$ was derived when $V(T, x, 1)$ is assumed to be smooth.

In Bouchard, Elie and Touzi (2009), a PDE characterization of $V(t, x, p)$ was derived.
- Assumptions: rather strong, e.g. existence of a unique strong solution of (1);
- main tool used: Geometric dynamic programming principle.

Under the No-Arbitrage condition, they recovered the solution of quantile hedging problem proposed in Follmer and Leukert (1999).
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In our paper, we will also work towards a PDE characterization for $V(t, x, p)$, but

- We only assume the existence of a weak solution of (1) that is unique in distribution;
- We admit arbitrage in our model
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Assumptions

Assumption M

Let $b_i : (0, \infty)^d \rightarrow \mathbb{R}$ and $s_{ik} : (0, \infty)^d \rightarrow \mathbb{R}$ be continuous functions and $b(\cdot) = (b_1(\cdot), \cdots, b_d(\cdot))'$ and $s(\cdot) = (s_{ij}(\cdot))_{1 \leq i,j \leq d}$, which we assume to be invertible for all $x \in (0, \infty)^d$.

We also assume that (1) has a weak solution that is unique in distribution for every initial value.

Let $\theta(\cdot) := s^{-1}(\cdot)b(\cdot)$, $a_{ij}(\cdot) := \sum_{k=1}^{d} s_{ik}(\cdot)s_{jk}(\cdot)$ s atisfy

$$\sum_{i}^{d} \int_{0}^{T} \left( |b_i(X(t))| + a_{ii}(X(t)) + \theta_i^2(X(t)) \right) < \infty.$$ (2)
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We denote by $\mathbb{F}$ the augmentation of the natural filtration of $X(\cdot)$.

Thanks to Assumption M,

- every local martingale of $\mathbb{F}$ has the martingale representation property with respect to $\mathcal{W}(\cdot)$ (adapted to $\mathbb{F}$).
- the solution of (1) takes values in the positive orthant
- the exponential local martingale

$$Z(t) := \exp \left\{ -\int_0^t \theta(X(s))'d\mathcal{W}(s) - \frac{1}{2} \int_0^t |\theta(X(s))|^2ds \right\},$$

(3)

the so-called *deflator* is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.
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the so-called deflator is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.
What does the existence of a deflator entail?

- While we do not assume the existence of equivalent local martingale measures, we assume the existence of a local martingale deflator (the $Z(\cdot)$ process). This is equivalent to the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition, introduced in Karatzas and Kardaras (2007).

- By Kardaras (2010), NUPBR is equivalent to the non-existence of arbitrages of the first kind, arbitrages that can be attained through nonegative wealth processes.

- So in our model, arbitrage may exist, but we cannot scale it up to make arbitrary amount of money.
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3. **THE PDE CHARACTERIZATION**
   - Stochastic Control Problem Formulation
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Let $g : (0, \infty)^d \to \mathbb{R}_+$ be a measurable function satisfying
\[
\mathbb{E}[Z(T)g(X(T))] < \infty.
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We want to determine
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for $p \in [0, 1]$.

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We will always assume Assumption M and (4) hold.
We will present a probabilistic characterization of $V(T, x, p)$.

**Lemma 3.1**

Given $A \in \mathcal{F}_T$,

(i) if $\mathbb{P}(A) \geq p$, then

$$V(T, x, p) \leq \mathbb{E}[Z(T)g(X(T))1_A].$$

(ii) if $\mathbb{P}(A) = p$ and

$$\text{ess sup}_A \{Z(T)g(X(T))\} \leq \text{ess inf}_{A^c} \{Z(T)g(X(T))\},$$

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Proof of Lemma 3.1

(i) Assumption M implies that the market is complete. So $Z(T)g(X^{t,x}(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$. Since $\mathbb{P}(A) \geq p$, it follows that $V(T, x, p) \leq \mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$.

(ii) Take arbitrary $y_0 > 0$ and $\pi_0 \in \mathcal{H}$ such that $\mathbb{P}\{B\} \geq p$, where $B \triangleq \{Y_{y_0,\pi_0}(T) \geq g(X(T))\}$.

To prove equality in (7), it’s enough to show that $y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A]$.

Observing that $\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$ and using (6), we obtain that...
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$$\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$$

and using (6), we obtain that
(I) Assumption M implies that the market is complete. So $Z(T)g(X^t,x(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X^t,x(T))1_A]$. Since $\mathbb{P}(A) \geq p$, it follows that $V(T,x,p) \leq \mathbb{E}[Z(T)g(X^t,x(T))1_A]$.

(II) take arbitrary $y_0 > 0$ and $\pi_0 \in \mathcal{H}$ such that

$$\mathbb{P}\{B\} \geq p, \text{ where } B \triangleq \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.$$ 

To prove equality in (7), it’s enough to show that

$$y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A].$$

Observing that

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$$

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Proof of Lemma 3.1

(I) Assumption M implies that the market is complete. So $Z(T)g(X_{t,x}(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X_{t,x}(T))1_A]$. Since $\mathbb{P}(A) \geq p$, it follows that $V(T,x,p) \leq \mathbb{E}[Z(T)g(X_{t,x}(T))1_A]$.

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Observing that

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and using (6), we obtain that
**Proof of Lemma 3.1**

(I) Assumption M implies that the market is complete. So \( Z(T)g(X^{t,x}(T))1_A \in \mathcal{F}_T \) is replicable with initial capital \( E[Z(T)g(X^{t,x}(T))1_A] \). Since \( P(A) \geq p \), it follows that \( V(T,x,p) \leq E[Z(T)g(X^{t,x}(T))1_A] \).

(II) take arbitrary \( y_0 > 0 \) and \( \pi_0 \in \mathcal{H} \) such that

\[
P\{B\} \geq p, \text{ where } B \triangleq \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.
\]

To prove equality in (7), it's enough to show that

\[
y_0 \geq E[Z(T)g(X(T))1_A].
\]

Observing that
\[
P(A^c \cap B) = P(A \cup B) - P(A) \geq P(A \cup B) - P(B) = P(B^c \cap A)
\]
and using (6), we obtain that
Proof of Lemma 3.1

(I) Assumption M implies that the market is complete. So $Z(T)g(X^{t,x}(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$. Since $\mathbb{P}(A) \geq p$, it follows that $V(T, x, p) \leq \mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$.

(II) take arbitrary $y_0 > 0$ and $\pi_0 \in \mathcal{H}$ such that

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To prove equality in (7), it’s enough to show that

$$y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A].$$

Observing that

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$$

and using (6), we obtain that
Proof of Lemma 3.1 (conti.)

\[ y_0 \geq \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)] \]
\[ = \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_B] + \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_{B^c}] \]
\[ \geq \mathbb{E}[Z(T)g(X(T))1_B] \]
\[ = \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A^c \cap B}] \]
\[ \geq \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{P}(A^c \cap B) \text{ess inf}_{A^c \cap B} \{Z(T)g(X(T))\} \]
\[ \geq \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{P}(A \cap B^c) \text{ess sup}_{A \cap B^c} \{Z(T)g(X(T))\} \]
\[ \geq \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A \cap B^c}] \]
\[ = \mathbb{E}[Z(T)g(X(T))1_A]. \]
Now we want show we can indeed find an $A \in \mathcal{F}_T$ satifying the coinditions in Lemma 3.1 (ii).

- Let $F(\cdot)$ be the cumulative distribution function of $Z(T)g(X(T))$.
- For any $a \in \mathbb{R}^+$ define
  
  $A_a := \{\omega : Z(T)g(X(T)) < a\}$, $\partial A_a := \{\omega : Z(T)g(X(T)) = a\}$,

  and let $\bar{A}_a$ denote $A_a \cup \partial A_a$.
- Taking $A = \bar{A}_a$ in Lemma 3.1, it follows that
  
  $$V(T, x, F(a)) = \mathbb{E}[Z(T)g(X(T))1_{\bar{A}_a}].$$

  (8)

  On the other hand, taking $A = A_a$, we obtain that

  $$V(T, x, F(a-)) = \mathbb{E}[Z(T)g(X(T))1_{A_a}].$$

  (9)

  The last two equalities imply the following relationship

  $$V(T, x, F(a)) = V(t, x, F(a-)) + a\mathbb{P}\{\partial A_a\}$$

  $$= V(t, x, F(a-)) + a(F(a) - F(a-)).$$

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The last two equalities imply the following relationship

\[
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- Taking $A = \bar{A}_a$ in Lemma 3.1, it follows that
  
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On the other hand, taking $A = A_a$, we obtain that

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The last two equalities imply the following relationship

$$V(T, x, F(a)) = V(t, x, F(a-)) + a\mathbb{P}\{\partial A_a\} = V(t, x, F(a-)) + a(F(a) - F(a-)).$$
Now we want show we can indeed find an $A \in \mathcal{F}_T$ satisfying the conditions in Lemma 3.1 (ii).

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Proposition 3.1

Next, we will determine $V(T, x, p)$ for $p \in (F(a-), F(a))$ when $F(a-) < F(a)$.

Proposition 3.1

Fix arbitrary $(t, x, p) \in (0, T) \times (0, \infty)^d \times [0, 1]$

(I) There exists $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (6). As a result, we have

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A].$$

(II) If $F^{-1}(p) := \{s \in \mathbb{R}_+: F(s) = p\} = \emptyset$, then letting $a := \inf\{s \in \mathbb{R}_+ : F(s) > p\}$ we have

$$V(T, x, p) = V(T, x, F(a-)) + a(p - F(a-)).$$

$$= V(T, x, F(a)) - a(F(a) - p)$$ (11)
Proposition 3.1

Next, we will determine $V(T, x, p)$ for $p \in (F(a-), F(a))$ when $F(a-) < F(a)$.

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Proof of Proposition 3.1

Assume $F^{-1}(p) := \{ s \in \mathbb{R}_+ : F(s) = p \} = \emptyset$. For (i),

- Let $\tilde{W}$ be a Brownian motion with respect to $\mathbb{F}$ and define $B_b = \{ \omega : \frac{\tilde{W}(T)}{\sqrt{T-t}} < b \}$.
- Define $f(\cdot)$ by $f(b) = \mathbb{P}\{ \partial A_a \cap B_b \}$. It satisfies
  
  $$\lim_{b \to -\infty} f(b) = 0 \text{ and } \lim_{b \to \infty} f(b) = \mathbb{P}(\partial A_a).$$

Moreover, it is continuous and nondecreasing. For continuity:

$$0 \leq f(b+\varepsilon) - f(b) = \mathbb{P}(\partial A_a \cap B_{b+\varepsilon}) - \mathbb{P}(\partial A_a \cap B_b) \leq \mathbb{P}(B_{b+\varepsilon} \cap B_b^c),$$

for $\varepsilon > 0 (< 0)$. As $\varepsilon \to 0$, one observes continuity.

- Since $0 < p - \mathbb{P}(A_a) < \mathbb{P}(\partial A_a)$, thanks to the above properties of $f$, there exists a $b^* \in \mathbb{R}_+$ satisfying $f(b^*) = p - \mathbb{P}(A_a)$.
- Define $A := A_a \cup (\partial A_a \cap B_{b^*})$. Observe that
  
  $$\mathbb{P}(A) = \mathbb{P}(A_a) + \mathbb{P}(\partial A_a \cap B_{b^*}) = p,$$

and $A$ satisfies (6).
Proof of Proposition 3.1

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Moreover, it is continuous and nondecreasing. For continuity:

\[ 0 \leq f(b + \varepsilon) - f(b) = \mathbb{P}(\partial A_a \cap B_{b+\varepsilon}) - \mathbb{P}(\partial A_a \cap B_b) \leq \mathbb{P}(B_{b+\varepsilon} \cap B_b^c) \]

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  $$

Moreover, it is continuous and nondecreasing. For continuity:

$$
0 \leq f(b+\varepsilon) - f(b) = \mathbb{P}(\partial A_a \cap B_{b+\varepsilon}) - \mathbb{P}(\partial A_a \cap B_b) \leq \mathbb{P}(B_{b+\varepsilon} \cap B_b^c),
$$

for $\varepsilon > 0(< 0)$. As $\varepsilon \to 0$, one observes continuity.

- Since $0 < p - \mathbb{P}(A_a) < \mathbb{P}(\partial A_a)$, thanks to the above properties of $f$, there exists a $b^* \in \mathbb{R}_+$ satisfying $f(b^*) = p - \mathbb{P}(A_a)$.

- Define $A := A_a \cup (\partial A_a \cap B_{b^*})$. Observe that
  
  $$
  \mathbb{P}(A) = \mathbb{P}(A_a) + \mathbb{P}(\partial A_a \cap B_{b^*}) = p, \text{ and } A \text{ satisfies (6)}.
  $$
For (ii), it follows immediately from (i),

\[ V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A] \]
\[ = \mathbb{E}[Z(T)g(X(T))1_{A_a}] + \mathbb{E}[Z(T)g(X(T))1_{\partial A_a \cap B_{b^*}}] \]
\[ = V(T, x, F(a^-)) + a\mathbb{P}(\partial A_a \cap B_{b^*}) \]
\[ = V(t, x, F(a^-)) + a(p - F(a^-)). \]
Remark

When $Z$ is a martingale:

- Using Neyman-Pearson Lemma, Follmer and Leukert (1999) showed that

$$ V(T, x, p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi] = \mathbb{E}[Z(T)g(X(T))\varphi^*], $$

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$$ \mathcal{M} = \{ \varphi : \Omega \to [0, 1] \text{ is } \mathcal{F}_T \text{ measurable s.t. } \mathbb{E}[\varphi] \geq p \}. $$

- The randomized test function $\varphi^*$ is not necessarily an indicator function. Using Lemma 3.1 and the fine structure of $\mathcal{F}_T$, in Proposition 3.1, we provide another optimizer of (12) that is an indicator function.
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Consider a market with a single stock, whose dynamics follow a three-dimensional Bessel process, i.e.

$$dX(t) = \frac{1}{X(t)} dt + dW(t) \quad X_0 = x > 0,$$

and let $g(x) = x$.

In this case, $Z(t) = x/X(t)$, which is the classical example for a strict local martingale; see Johnson and Helms (1963). On the other hand, $Z(t)X(t) = x$ is a martingale.

Thanks to Proposition 3.1 there exits a set $A \in \mathcal{F}_T$ with $\mathbb{P}(A) = p$ such that

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Here, we will give alternative representation of $V$, which facilitates its PDE characterization in the next section. Recall that

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Proof of Proposition 3.3

Thanks to Proposition 3.1 there exists a set $A \in \mathcal{F}_T$ such that $V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A]$. Since $1_A \in \mathcal{M}$, clearly

$$V(T, x, p) \geq \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

For the other direction, we will show that for any $\varphi \in \mathcal{M}$ and a given set $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (6), we have

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Proof of Proposition 3.3 (cont.)

Letting $M = \text{ess sup}_A \{ Z(T) g(X(T)) \}$, we can write

\[
\begin{align*}
\mathbb{E}[Z(T)g(X(T))\varphi] - \mathbb{E}[Z(T)g(X(T))1_A] \\
= \mathbb{E}[Z(T)g(X(T))\varphi 1_A] + \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] \\
- \mathbb{E}[Z(T)g(X(T))1_A] \\
= \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A(1 - \varphi)] \\
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OUTLINE

1. **INTRODUCTION**
   - The Problem
   - Related Work
   - The Model

2. **ON QUANTILE HEDGING**

3. **THE PDE CHARACTERIZATION**
   - Stochastic Control Problem Formulation
   - Associated PDE
Let us denote by $P^p_\alpha(\cdot)$ the solution of

$$dP(t) = P(t)(1 - P(t))\alpha'(t)dW(t), \quad P(0) = p \in [0, 1],$$

where $\alpha(\cdot)$ is an $\mathbb{F}$—progressively measurable $\mathbb{R}^d$-valued process such that $\int_0^T \|\alpha(s)\|^2 ds < \infty \mathbb{P}$-a.s. We will denote the class of such processes by $\mathcal{A}$.

The next result obtains an alternative representation for $V$ in terms of $P$.

**Proposition 4.1**

$$V(T, x, p) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z(T)g(X(T))P^p_\alpha(T)] < \infty.$$
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The finiteness follows from (4).

It can be shown using Proposition 3.3 that

\[ V(T, x, p) = \inf_{\varphi \in \tilde{M}} \mathbb{E}[Z(T)g(X(T))\varphi], \]

where \( \tilde{M} = \{ \varphi : \Omega \rightarrow [0, 1] \text{ is } \mathcal{F}_T \text{ measurable s.t. } \mathbb{E}[\varphi] = p \} \).

Therefore it’s enough to show that \( \tilde{M} \) satisfies

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The inclusion \( \widetilde{M} \supset \{ P_\alpha^p(T) : \alpha \in \mathcal{A} \} \) is clear. To show the other inclusion, use the martingale representation theorem: For any \( \varphi \in \mathcal{F}_T \) there exists an \( \mathbb{F} \)-progressively measurable \( \mathbb{R}^d \)-valued process \( \psi(\cdot) \) satisfying

\[
\mathbb{E}[\varphi|\mathcal{F}_t] = p + \int_0^t \psi'(s)dW(s).
\]

Then we see that \( \mathbb{E}[\varphi|\mathcal{F}_t] \) solves (14) with \( \alpha(\cdot) \)

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\alpha(t) = 1\{\mathbb{E}[\varphi|\mathcal{F}_t] \in (0,1)\} \cdot \frac{\psi(t)}{\mathbb{E}[\varphi|\mathcal{F}_t](1 - \mathbb{E}[\varphi|\mathcal{F}_t])}.
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The inclusion $\tilde{\mathcal{M}} \supset \{ P^p_\alpha(T) : \alpha \in \mathcal{A} \}$ is clear. To show the other inclusion, use the martingale representation theorem: For any $\varphi \in \mathcal{F}_T$ there exists an $\mathbb{F}$—progressively measurable $\mathbb{R}^d$-valued process $\psi(\cdot)$ satisfying

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The inclusion $\tilde{M} \supset \{P^p_\alpha(T) : \alpha \in \mathcal{A}\}$ is clear. To show the other inclusion, use the martingale representation theorem: For any $\varphi \in \mathcal{F}_T$ there exists an $\mathbb{F}$—progressively measurable $\mathbb{R}^d$-valued process $\psi(\cdot)$ satisfying

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We denote by $X^{t,x}(\cdot)$ the solution of (1) starting from $x$ at time $t$ and by $P_{\alpha}^{t,p}(\cdot)$ the solution of (14) starting from $p$ at time $t$. We also introduce $Z^{t,x,z}(\cdot)$ as the solution of

$$dZ(s) = -Z(s)\theta(X^{t,x}(s))'dW(s), \quad Z(t) = z,$$

(15)

and the value function

$$U(t, x, p) := \inf_{\alpha \in A} \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))P_{\alpha}^{t,p}(T)].$$

(16)

the original value function $V$ can be written in terms of $U$ as

$$V(T, x, p) = U(0, x, p).$$
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The Value Function $U$

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\[
dZ(s) = -Z(s)\theta(X^{t,x}(s))'dW(s), \quad Z(t) = z, \quad (15)
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U(t, x, p) := \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z^{t,x,1\left(T\right)}g(X^{t,x}(T))P^{t,p}_{t,\alpha}(T)]. \quad (16)
\]

- the original value function \( V \) can be written in terms of \( U \) as

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\]
First define

$$\Lambda(t, \cdot) := \frac{x_1 + \cdots + x_d}{Z^{t,x,1}(\cdot)(X_1^{t,x}(\cdot) + \cdots + X_d^{t,x}(\cdot))}$$

$$= \exp \left( \int_t^\cdot \left( \tilde{\theta}(X^{t,x}(u)) \right)' d\tilde{W}(u) - \frac{1}{2} \int_t^\cdot \|\tilde{\theta}(X^{t,x}(u))\|^2 du \right)$$

in which \(\tilde{\theta}(\cdot) := \theta(\cdot) - s'(\cdot)m(\cdot)\), where \(m\) is defined by

$$m_i(x) = \frac{x_i}{x_1 + \cdots + x_d}, \quad i = 1, \cdots, d,$$

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\[ \tilde{W}(s) := W(s) + \int_{t}^{s} \tilde{\theta}(X(u)) du, \quad s \geq t. \]
First define

\[ \Lambda(t, \cdot) := \frac{x_1 + \cdots + x_d}{Z_{t,x,1}(\cdot)(X_{1,t}^{x}(\cdot) + \cdots + X_{d,t}^{x}(\cdot))} \]

\[ = \exp \left( \int_t^\cdot (\tilde{\theta}(X_{t}^{x}(u)))' \tilde{W}(u) - \frac{1}{2} \int_t^\cdot \|\tilde{\theta}(X_{t}^{x}(u))\|^2 du \right) \]

in which \( \tilde{\theta}(\cdot) := \theta(\cdot) - s'(\cdot)m(\cdot) \), where \( m \) is defined by

\[ m_i(x) = x_i / (x_1 + \cdots + x_d), \quad i = 1, \cdots, d, \]

and

\[ \tilde{W}(s) := W(s) + \int_t^s \tilde{\theta}(X(u))du, \quad s \geq t. \]
**Express \( U \) Under A New Measure \( Q \)**

- First define

\[
\Lambda(t, \cdot) := \frac{x_1 + \cdots + x_d}{Z^{t,x,1}(\cdot)(X^{t,x}_{1,\cdot}(\cdot) + \cdots + X^{t,x}_{d,\cdot}(\cdot))}
\]

\[
= \exp \left( \int_t^\cdot (\tilde{\theta}(X^{t,x}(u)))' d\tilde{W}(u) - \frac{1}{2} \int_t^\cdot \|\tilde{\theta}(X^{t,x}(u))\|^2 du \right)
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\]

and

\[
\tilde{W}(s) := W(s) + \int_t^s \tilde{\theta}(X(u)) du, \quad s \geq t.
\]
There exists a probability measure \( \mathbb{Q} \) on \( (\Omega, \mathcal{F}) \) such that 
\[
d\mathbb{P} = \Lambda(t, T) d\mathbb{Q}
\]
on each \( \mathcal{F}(T) \), for \( T \in (t, \infty) \). Under \( \mathbb{Q} \),
\( \tilde{W}(\cdot) \) is a Brownian motion and we have that
\[
\mathbb{E}[Z_{t,x,1}(T)(X_{1,t,x}(T) + \cdots + X_{d,t,x}(T))] = \mathbb{Q}(T > t),
\]
for all \( T \in [0, \infty) \), where
\[
T = \inf \left\{ s \geq t : \int_t^s \| \tilde{\theta}(X_{t,x}(u)) \|^2 du = \infty \right\}.
\]
Express $U$ under a new measure $Q$ (conti.)

There exists a probability measure $Q$ on $(\Omega, \mathcal{F})$ such that $d\mathbb{P} = \Lambda(t,T)dQ$ on each $\mathcal{F}(T)$, for $T \in (t, \infty)$. Under $Q$, $\tilde{W}(\cdot)$ is a Brownian motion and we have that

$$E[Z^{t,x,1}(T)(X^t_1(x)(T) + \cdots + X^t_d(x)(T))] = Q(T > t),$$

for all $T \in [0, \infty)$, where

$$T = \inf \left\{ s \geq t : \int_t^s \|\tilde{\theta}(X^t,x(u))\|^2 du = \infty \right\}.$$
Express \( U \) under a new measure \( Q \) (conti.)

- There exists a probability measure \( Q \) on \((\Omega, \mathcal{F})\) such that 
  \[d\mathbb{P} = \Lambda(t, T)dQ\] on each \( \mathcal{F}(T) \), for \( T \in (t, \infty) \). Under \( Q \), \( \tilde{W}(\cdot) \) is a Brownian motion and we have that 
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  \]
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for all $T \in [0, \infty)$, where
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T = \inf \left\{ s \geq t : \int_{t}^{s} \| \tilde{\theta}(X_{t, x}(u)) \|^2 du = \infty \right\}.
\]
We will make the following assumption to obtain a representation of $\mathcal{T}$ in terms of $X$.

**Assumption 4.1**

$$\|\theta\|^2 \leq C(1 + \text{Trace}(a)).$$

Under this assumption, it follows $\mathcal{Q}$–a.e. that

$$\mathcal{T} = \min_{1 \leq i \leq d} \mathcal{T}_i, \quad \text{in which} \quad \mathcal{T}_i = \inf\{s \geq t : X_{i,t,x}(s) = 0\}.$$

For these claims about the existence and the properties of the probability measure $\mathcal{Q}$ see Fernholz and Karatzas (2008, 2010), and the references therein.
We will make the following assumption to obtain a representation of $T$ in terms of $X$.

**Assumption 4.1**

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Under this assumption, it follows $Q$–a.e. that

$$T = \min_{1 \leq i \leq d} T_i,$$

in which

$$T_i = \inf\{s \geq t : X_{t}^{i,x}(s) = 0\}.$$

For these claims about the existence and the properties of the probability measure $Q$ see Fernholz and Karatzas (2008, 2010), and the references therein.
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For these claims about the existence and the properties of the probability measure $\mathbb{Q}$ see Fernholz and Karatzas (2008, 2010), and the references therein.
Now, $U$ can be represented in terms of $Q$ as

$$U(t, x, p) = (x_1 + \cdots + x_d) \inf_{\alpha \in A} \mathbb{E}^Q \left[ \frac{g(X^{t,x}(T))}{X_1^{t,x}(T) + \cdots + X_d^{t,x}(T)} P^t_p(T) 1\{T > T\} \right].$$
Now, $U$ can be represented in terms of $\mathbb{Q}$ as

$$U(t, x, p) = (x_1 + \cdots + x_d) \inf_{\alpha \in \mathcal{A}} \mathbb{E}^\mathbb{Q} \left[ \frac{g(X_{t,x}^T)}{X_{1,t,x}(T) + \cdots + X_{d,t,x}(T)} P_{t,p}^\alpha(T) 1\{T > T\} \right].$$
The dynamics of $X_{i}^{t,x}(s)$ and $P_{i}^{t,p}(s)$ in terms of the $Q$-Brownian motion $\tilde{W}$ can be written as

$$dX_{i}^{t,x}(s) = X_{i}^{t,x}(s)\left(\sum_{j=1}^{d} a_{ij}(X_{j}^{t,x}(s)) X_{j}^{t,x}(s) / X_{1}^{t,x}(s) + \cdots + X_{d}^{t,x}(s) / X_{d}^{t,x}(s)\right) \, ds + \sum_{k=1}^{d} s_{ik}(X_{i}^{t,x}(s)) d\tilde{W}_{k}(s),$$

for $i = 1, \cdots, d$, and

$$dP_{i}^{t,p}(s) = P_{i}^{t,p}(s)(1 - P_{i}^{t,p}(s)) \alpha'(s)\left(-\tilde{\theta}(X_{i}^{t,x}) ds + d\tilde{W}(s)\right).$$ (17)
The dynamics of $X^{t,x}$ and $P^{t,p}$ in terms of the $\mathbb{Q}$-Brownian motion $\tilde{W}$ can be written as

$$dX_i^{t,x}(s) = X_i^{t,x}(s) \left( \frac{\sum_{j=1}^{d} a_{ij}(X^{t,x}(s)) X_j^{t,x}(s)}{X_1^{t,x}(s) + \cdots + X_d^{t,x}(s)} ds + \sum_{k=1}^{d} s_{ik}(X^{t,x}(s)) d\tilde{W}_k(s) \right),$$

for $i = 1, \cdots, d$, and

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for $i = 1, \cdots, d$, and

$$dP^{t,p}(s) = P^{t,p}(s)(1 - P^{t,p}(s))\alpha'(s)(-\tilde{\theta}(X^{t,x}) ds + d\tilde{W}(s)). \quad (17)$$
To apply the dynamic programming principle due to Haussmann and Lepeltier (1990), we assume

**Assumption 4.2**

For all $y \in \mathbb{R}_+^d - \{0\}$ we have the following growth condition

$$\sum_{i=1}^{d} \sum_{j=1}^{d} y_i y_j |a_{ij}(y)| \leq C(1 + \|y\|).$$

for some constant $C$.

**Assumption 4.3**

The mapping $(t, x, p) \rightarrow \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))P_{\alpha}^{t,p}(T)]$ is lower semi-continuous on $t \in [0, T], x \in \mathbb{R}_+^d, p \in [0, 1]$, for all $\alpha \in \mathcal{A}$.
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**Assumption 4.3**

The mapping $(t, x, p) \rightarrow \mathbb{E}[Z_{t,x}^1(T) g(X_{t,x}^1(T)) P_{t,p}^T(T)]$ is lower semi-continuous on $t \in [0, T], x \in \mathbb{R}_+^d, p \in [0, 1]$, for all $\alpha \in \mathcal{A}$. 

Yu-Jui Huang

Outperforming The Market Portfolio With A Given Probability
Proposition 4.2

Under Assumption M, 4.1, 4.2 and 4.3,

(1) $U^*$ is a viscosity subsolution of

$$
\partial_t U^* + \frac{1}{2} \text{Trace} \left( \sigma \sigma' D_x^2 U^* \right) + \inf_{a \in \mathbb{R}^d} \left\{ (D_{xp} U^*)' \sigma a + \frac{1}{2} |a|^2 D_p^2 U^* - \theta' a D_p U^* \right\} \geq 0,
$$

with the boundary conditions

$U^*(t, x, 1) = \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))], \ U^*(t, x, 0) = 0,$

and

$U^*(T, x, p) \leq pg(x).$
Proposition 4.2

Under Assumption M, 4.1, 4.2 and 4.3,

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$U^*(t, x, 1) = \mathbb{E}[Z^{t,x,1}(T) g(X^{t,x}(T))]$, $U^*(t, x, 0) = 0$, and $U^*(T, x, p) \leq pg(x)$. 

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Outperforming The Market Portfolio With A Given Probability
**Proposition 4.2 (conti.)**

(II) $U^*$ is a viscosity supersolution of

$$
\begin{align*}
\frac{\partial_t}{t} U_* + \frac{1}{2} \text{Trace} \left( \sigma \sigma' D_x^2 U_* \right) \\
+ \inf_{a \in \mathbb{R}^d} \left\{ (D_x p U_*)' \sigma a + \frac{1}{2} |a|^2 D_p^2 U_* - \theta' a D_p U_* \right\}
\leq 0
\end{align*}
$$

(18)

with the boundary conditions

$U^*(t, x, 1) = \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))], U^*(t, x, 0) = 0,$ and $U^*(T, x, p) \geq pg(x)$.
Remark

Let us consider the PDE satisfied by the superhedging price $U(t, x, 1)$:

$$0 = v_t + \frac{1}{2} \text{Tr}(\sigma \sigma' D^2_x v), \quad \text{on } (0, T) \times (0, \infty)^d, \quad (19)$$

$$v(T-, x) = g(x), \quad \text{on } (0, \infty)^d. \quad (20)$$

Unless additional boundary conditions are specified, this PDE may have multiple solutions, see e.g. the volatility stabilized model of Fernholz and Karatzas (2008). Even when additional boundary conditions are specified, the growth of $\sigma$ might lead to the loss of uniqueness. In the one-dimensional case one can determine an explicit condition which is sufficient and necessary for uniqueness (non-uniqueness), see e.g. Bayraktar and Xing (2010).
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Let us consider the PDE satisfied by the superhedging price $U(t, x, 1)$:

$$0 = v_t + \frac{1}{2} Tr(\sigma \sigma' D_x^2 v), \quad \text{on } (0, T) \times (0, \infty)^d,$$  \hspace{1cm} (19)

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Unless additional boundary conditions are specified, this PDE may have multiple solutions, see e.g. the volatility stabilized model of Fernholz and Karatzas (2008). Even when additional boundary conditions are specified, the growth of $\sigma$ might lead to the loss of uniqueness. In the one-dimensional case one can determine an explicit condition which is sufficient and necessary for uniqueness (non-uniqueness), see e.g. Bayraktar and Xing (2010).
Let $\Delta U(t, x, 1)$ be the difference of two solutions of (19)-(20). Then both $U(t, x, p)$ and $U(t, x, p) + \Delta U(t, x, 1)$ are viscosity supersolution of (18) (along with its boundary conditions). As a result when (19) and (20) has multiple solutions so does the PDE for the function $U$. 
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Thank you very much for your attention!

Q & A