Quasi-Sure Analysis: from Model-Free Hedging to 2BSDE

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Outline

The Standard Super-Hedging Problem

- $\Omega = \{\omega \in C(\mathbb{R}_+) : \omega(0) = 0\}$,
- B coordinate process, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, $\mathbb{P}_0$: Wiener measure

- Zero interest rate, and risky asset defined by :

$$dS_t = S_t \sigma_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

for some $\mathbb{F}-$prog. meas. σ with $\int_0^T |\sigma_t|^2 dt < \infty$, $\mathbb{P}_0-$a.s.

- Super-hedging problem of $\mathcal{F}_T-$meas. r.v. ξ

$$U_0 := \inf\left\{X_0 : \ X_0 + \int_0^T H_t \sigma_t dB_t \geq \xi, \ \mathbb{P}_0 - \text{a.s. for some } H \in \mathcal{H}_0\right\}$$

Then $U_0 = \mathbb{E}^{\mathbb{P}_0}[\xi]$, existence holds, and perfect replication

Unspecified Volatility Process

Suppose that B is only known to be a local martingale with quadratic variation $\langle B\rangle$ a.c. wrt Lebesgue. Let

$$\mathcal{P} := \left\{\mathbb{P}_0 \circ \left(\int_0^\cdot \sigma_t dB_t\right)^{-1} : \ \sigma \ \mathbb{F}-\text{prog. meas.}, \ \int_0^T |\sigma_t|^2 dt < \infty\right\}$$

• Super-hedging and Sub-hedging problems of $\mathcal{F}_T-$meas. r.v. ξ

$$U := \inf\left\{X_0 : \ \exists\, H \in \mathcal{H} : X_0 + \int_0^T H_t\sigma_t dB_t \geq \xi, \ \mathcal{P} - \text{q.s.}\right\}$$

$$L := \sup\left\{X_0 : \ \exists\, H \in \mathcal{H} : X_0 + \int_0^T H_t\sigma_t dB_t \leq \xi, \ \mathcal{P} - \text{q.s.}\right\}$$

• $\mathcal{P}-$q.s. means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

• the portfolio $H \in \mathcal{H}$ does not depend on a particular $\mathbb{P}$...

Dual Formulation

Expect to have :

$$U = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad L = \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$$

and existence...

- Denis-Martini 2005
- Peng 2007 for the bounded volatility case
- Soner, T. Zhang 2010 for the unbounded volatility case

Application to Model-free bounds

• Suppose that prices of $T-$maturity call options for all possible strikes $c(k), k \geq 0$ are observed and tradable, i.e. $S_T \sim_{\mathbb{P}} \mu$

• Then, the no-arbitrage bounds can be improved to

$$U(\mu) := \inf\left\{X_0 : \exists\, H \in \mathcal{H},\ \lambda \in \Lambda : X_T^{H,\lambda} \geq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$
$$L(\mu) := \sup\left\{X_0 : \exists\, H \in \mathcal{H},\ \lambda \in \Lambda : X_T^{H,\lambda} \leq \xi,\ \mathcal{P}-\text{q.s.}\right\}$$

where $X_T^{H,\lambda} := X_0 + \int_0^T H_t dB_t + \int \lambda(k)[(S_T-k)^+ - c(k)]dk$

• Alternatively, one may formulate the problems :

$$\overline{U}(\mu) := \sup_{\mathbb{P}\in\mathcal{P}: B_T \sim_{\mathbb{P}} \mu} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad \overline{L}(\mu) := \inf_{\mathbb{P}\in\mathcal{P}: B_T \sim_{\mathbb{P}} \mu} \mathbb{E}^{\mathbb{P}}[\xi]$$

Optimal transportation problem...

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\begin{aligned}
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Optimal transportation problem...

Duality and stochastic control

- The previous duality implies that

$$U(\mu) := \inf_{\lambda\in\Lambda} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi - \int \lambda(k)\big((S_T - k)^+ - c(k)\big)\Big]$$

$$L(\mu) := \sup_{\lambda\in\Lambda} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi - \int \lambda(k)\big((S_T - k)^+ - c(k)\big)\Big]$$

References :
P. Carr, A. Cox, M. Davis, B. Dupire, D. Hobson, R. Lee, J. Obloj
Galichon, Henry-Labordère, T.

Outline

Peng's Nonlinear G-Expectation

- Bounded volatility : $\sigma_.^2 \in [\underline{a}, \bar{a}]$
- Consider the nonlinear PDE

$$-\partial_t v - G(D^2 v) = 0 \quad \text{where} \quad G(\gamma) := \frac{1}{2} \sup_{\underline{a} \le a \le \bar{a}} \mathrm{Tr}[a\gamma]$$

- For $\xi \in \mathrm{Lip} := \{\phi(B_{t_1}, \ldots, B_{t_n}),\ \phi \text{ Lip}, n \in \mathbb{N}\}$, define

$\mathbf{E}^G[\xi]$ by solving backward the PDE in each interval $[t_i, t_{i+1}]$

- Define $\mathbf{E}^G$ on the closure $\mathcal{L}_G^2$ of Lip wrt $\|.\|_{\mathcal{L}_G^2} := \sup_{\mathbb{P}\in\mathcal{P}} \|.\|_{\mathbb{L}^2(\mathbb{P})}$

Then, by classical stochastic control theory :

$$\mathbf{E}^G[\xi] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$$

Duality and Existence under Bounded Volatility

Let $\mathcal{L}^2_{\mathcal{P}}$ be the closure of Lip wrt to :

$$\|\xi\|_{\mathcal{L}^2_{\mathcal{P}}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} \operatorname{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[|\xi|^2]\Big]$$

and consider the superhedging problem

$$U_0 := \inf\left\{X_0 : \; X_0 + \int_0^T H_t\sigma_t dB_t \geq \xi, \; \mathbb{P}_0 - \text{a.s. for some } H \in \mathcal{H}_0\right\}$$

where

$$\mathcal{H}_0 := \left\{H : \; H \in \mathbb{H}^2_{loc}(\mathbb{P}) \text{ and } X^H \geq \text{Mart}^{\mathbb{P}}, \; \forall \mathbb{P} \in \mathcal{P}\right\}$$

Theorem (Soner, Zhang, T.) For all $\xi \in \mathcal{L}^2_{\mathcal{P}}$:

$$U_0 = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$$

and existence holds for the problem U_0

The Unbounded Volatility Case

Repeat Peng's construction by considering

- Lip:$= \{\phi(B_{t_i}, \langle B\rangle_{t_i}, 1 \le i \le N),\ \phi$ Lipschitz, $n \in \mathbb{N}\}$
- the nonlinear PDE $\min\{v-\phi, 2v_y + v_{xx}\} = 0$

• For $\xi \in$ Lip , define

$\mathbf{E}[\xi]$ by solving backward the PDE in each interval $[t_i, t_{i+1}]$

• Define $\mathbf{E}$ on the closure $\mathcal{L}^2_{\mathcal{P}}$ of Lip wrt

$$\|\xi\|_{\mathcal{L}^2_{\mathcal{P}}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} \operatorname{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[|\xi|^2]\Big]$$

Theorem (Galichon, Henry-Labordère, T.) For all $\xi \in \mathcal{L}^2_{\mathcal{P}}$, $U_0 = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$, and existence holds for the problem U_0

Outline

BSDEs

- Find $(Y,Z) \in \mathcal{S}^2(\mathbb{P}_0) \times \mathbb{H}^2(\mathbb{P}_0)$ solution of

$$Y_t \;=\; \xi + \int_t^T f_s(Y_s,Z_s)ds - \int_t^T Z_s dB_s,\; \mathbb{P}_0 - \text{a.s.}$$

- In the Markov case $\xi = g(X_T)$ and $f_s(y,z) = F(s,X_s,y,z)$ for some Markov X defined by

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s$$

we have $Y_t = V(t,X_t)$, and the function V solves the semilinear PDE

$$\partial_t V + \mu \cdot DV + \frac{1}{2}\text{Tr}[\sigma\sigma^{\mathrm{T}} D^2 V] + F(.,V,\sigma^{\mathrm{T}} DV) \;=\; 0$$

Intuition for general fully nonlinear PDEs

Consider the 2BSDE

$$Y_t = g(W_T) + \int_t^T h(s, W_s, Y_s, Z_s, \Gamma_s)ds - \int_t^T Z_s \circ dW_s$$

and the corresponding PDE :

$$\partial_t V + h(t, x, V, DV, D^2 V) = 0, \;\; V(T, x) = g(x)$$

- If $h(t, x, y, z, \gamma) = \sup_{a \in A}[\frac{1}{2} a\gamma - F(t, x, y, z, a)]$, then $V = \sup_a V^a$:

$$\partial_t V^a + \frac{1}{2} a D^2 V^a - F(t, x, V^a, DV^a, a) = 0, \quad V^a(1, x) = g(x)$$

- Consequently, $Y_0 = \sup_a Y_0^a$, where
  $X_t^a = \int_0^t a_s^{1/2} dW_s$;
  $Y_t^a = g(X_1^a) - \int_t^1 F(s, X_s^a, Y_s^a, Z_s^a, a_s)ds - \int_t^1 Z_s^a dX_s^a$.

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Nonlinear generators

$H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$

$D_H \subset \mathbb{R}^{d\times d}$ given, containing 0

- For fixed (y, z, γ), H is $\mathbb{F}-$progressively measurable
- H is uniformly Lipschitz in (y, z), lsc in γ
- H is uniformly continuous in ω under the $\mathbb{L}^\infty-$norm

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\mathrm{Tr}[a\gamma] - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R})$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \text{ and } \hat{F}_t^0 := \hat{F}_t(0, 0)$$

For $1 < \kappa \le 2$, define $\mathcal{P}_H^\kappa$ as the set of all $\mathbb{P} \in \mathcal{P}$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \text{ for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \text{ and } \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^1 |\hat{F}_t^0|^\kappa dt\right)^{2/\kappa}\right] < \infty.$$

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Definition

For $\mathcal{F}_T-$meas. ξ, consider the 2BSDE :

$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t, \; 0 \le t \le 1, \; Y_T = \xi, \; \mathcal{P}_H^\kappa - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H^\kappa-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H^\kappa$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \; t \in [0,1], \; \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ satisfies :

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}'}], \; \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H^\kappa, t \le T$$

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Back to standard BSDEs

For standard BSDEs

$$dY_t := -H_t^0(Y_t, Z_t)dt + Z_t dB_t,\;\; Y_T = \xi$$

the nonlinearity and the corresponding conjugate :

$$H_t(.,\gamma) := \frac{-1}{2}\mathrm{Tr}[\gamma] + H_t^0(.),\quad F_t(.,a) = \begin{cases} -H_t^0(.) & \text{for } a = I_d \\ \infty & \text{otherwise} \end{cases}$$

Then $\mathcal{P}_H^\kappa = \{\mathbb{P}^0\}$, $K^{\mathbb{P}^0} \equiv 0$, and the previous definition reduces to the standard one

$$dY_t := -H_t^0(Y_t, Z_t)dt + Z_t dB_t,\;\; Y_T = \xi,\;\; \mathbb{P}^0 - \text{a.s.}$$

Wellposedness of 2BSDEs

$\mathcal{L}^2_{H,\kappa}$:= closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|.\|_{\mathcal{L}^2_{H,\kappa}}$:

$$\|\xi\|_{\mathcal{L}^2_H} \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T} \big(\mathbb{E}^{H,\mathbb{P}}_t\big[|\xi|^\kappa\big]\big)^{\frac{2}{\kappa}}\Big],$$

and

$$\mathbb{E}^{H,\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^\kappa_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

Theorem (Soner, Zhang, T. '10) For any $\xi \in \mathcal{L}^2_{H,\kappa}$, the 2BSDE admits a unique solution $(Y, Z) \in \mathbb{D}^2_{H,\kappa} \times \mathbb{H}^2_{H,\kappa}$

Connection with PDEs

Theorem (Soner, Zhang, T. '10) Under "natural conditions", the solution of the 2BSDE satisfies $Y_t = u(t, B_t)$, $t \in [0, T]$, $\mathcal{P}_H^\kappa$−q.s. and u is a viscosity solution of

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \hat{H}\Big(t, x, u(t,x), Du(t,x), D^2u(t,x)\Big) &= 0, \ 0 \le t < 1 \\ u(1,x) = g(x) & \end{aligned}$$

where

$$\hat{H}(t,x,y,z,\gamma) = \sup_{a \in \mathbb{S}_d^+(\mathbb{R})} \Big\{\frac{1}{2}\mathrm{Tr}[a\gamma] - F(t,x,y,z,a)\Big\}, \ \gamma \in \mathbb{R}^{d\times d}.$$

We also have a Feynman-Kac representation theorem for the Cauchy problem with the latter fully nonlinear PDE