Optimal Stopping for Dynamic Convex Risk Measures

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A joint work with

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Outline

1. Dynamic Convex Risk Measures (DCRMs)
2. A Robust Representation of DCRMs
3. Robust Optimal Stopping
4. Saddle Point Problem

Dynamic Convex Risk Measures

- **B** — a d-dimenional Brownian Motion on a probability space $(\Omega, \mathcal{F}, P)$
- $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ — an augmented filtration generated by **B**.
- $\nu \leq \gamma$ — two **F**-stopping times ν, γ with $\nu \leq \gamma$, P-a.s.
- $\mathcal{S}_{\nu,\gamma} \stackrel{\triangle}{=} \big\{ \mathbf{F}\text{-stopping times } \sigma : \ \nu \leq \sigma \leq \gamma, \ P\text{-a.s.} \big\}$

A **dynamic convex risk measure** (DCRM) is a family of mappings $\big\{\rho_{\nu,\gamma} : \mathbb{L}^\infty(\mathcal{F}_\gamma) \to \mathbb{L}^\infty(\mathcal{F}_\nu)\big\}_{\nu \leq \gamma}$ such that $\forall\, \xi, \eta \in \mathbb{L}^\infty(\mathcal{F}_\gamma)$

- **"Monotonicity":** $\rho_{\nu,\gamma}(\xi) \leq \rho_{\nu,\gamma}(\eta)$, P-a.s. if $\xi \geq \eta$, P-a.s.
- **"Translation Invariance":** $\rho_{\nu,\gamma}(\xi + \eta) = \rho_{\nu,\gamma}(\xi) - \eta$, P-a.s. if $\eta \in \mathbb{L}^\infty(\mathcal{F}_\nu)$.
- **"Convexity":** $\forall\, \lambda \in (0,1)$
  $\rho_{\nu,\gamma}\big(\lambda\xi + (1-\lambda)\eta\big) \leq \lambda\rho_{\nu,\gamma}(\xi) + (1-\lambda)\rho_{\nu,\gamma}(\eta)$, P-a.s.
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Assumptions

(A1) "Continuity from above": If $\xi_n \searrow \xi$ in $\mathbb{L}^\infty(\mathcal{F}_\gamma)$, then

$$\lim_{n\to\infty} \uparrow \rho_{\nu,\gamma}(\xi_n) = \rho_{\nu,\gamma}(\xi), \quad P\text{-a.s.}$$

(A2) "Time Consistency": $\rho_{\nu,\sigma}(-\rho_{\sigma,\gamma}(\xi)) = \rho_{\nu,\gamma}(\xi)$, P-a.s., $\forall\, \sigma \in \mathcal{S}_{\nu,\gamma}$.

(A3) "Zero-One Law": $\rho_{\nu,\gamma}(\mathbf{1}_A \xi) = \mathbf{1}_A\, \rho_{\nu,\gamma}(\xi)$, P-a.s., $\forall\, A \in \mathcal{F}_\nu$.

(A4) $\underset{\xi\in\mathcal{A}_t}{\operatorname{essinf}}\, E_P\left[\xi|\mathcal{F}_t\right] = 0$, where $\mathcal{A}_t \stackrel{\triangle}{=} \{\xi \in \mathbb{L}^\infty(\mathcal{F}_T) : \rho_{t,T}(\xi) \le 0\}$.

Example

- Entropic Risk Measure:

$$\rho^{\alpha}_{\nu,\gamma}(\xi) = \alpha \ln \left\{ E\left[e^{-\frac{1}{\alpha}\xi} \middle| \mathcal{F}_\nu \right] \right\}, \quad \xi \in \mathbb{L}^\infty(\mathcal{F}_\gamma)$$

is a *DCRM* satisfying (A1)-(A4).

($\alpha > 0$ is referred to as "*risk tolerance coefficient*").

Motivation

Optimal Stopping for DCRMs

Given $\nu \in \mathcal{S}_{0,T}$ and a bounded, adapted reward process Y we are interested in finding a stopping time $\tau_*(\nu) \in \mathcal{S}_{\nu,T}$ such that

robust O.S.

$$\rho_{\nu,\tau_*(\nu)}\left(Y_{\tau_*(\nu)}\right) = \underset{\gamma \in \mathcal{S}_{\nu,T}}{\operatorname{essinf}} \rho_{\nu,\gamma}\left(Y_\gamma\right), \qquad P\text{-a.s.} \tag{1}$$

Robust Representation of DCRMs

- $\mathcal{M}^e$ — the set of all probability measures on $(\Omega, \mathcal{F})$ that are equivalent to P.
- $\forall\, Q \in \mathcal{M}^e$, its density process Z^Q w.r.t. P is the stochastic exponential of some process θ^Q with $\int_0^T |\theta_t^Q|^2 dt < \infty$, P-a.s.

$$Z_t^Q = \mathscr{E}\left(\theta^Q \bullet B\right)_t = \exp\left\{\int_0^t \theta_s^Q dB_s - \frac{1}{2}\int_0^t \left|\theta_s^Q\right|^2 ds\right\}.$$

- $\forall\, \nu \in \mathcal{S}_{0,T}$ we define

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A Robust Representation of DCRMs (Delbaen, Peng and Rosazza-Gianin, '09)

Let $\{\rho_{\nu,\gamma}\}_{\nu\leq\gamma}$ be a DCRM satisfying (A1)-(A4). $\forall\,\nu\leq\gamma$ and $\xi\in\mathbb{L}^\infty(\mathcal{F}_\gamma)$, we have (1)

$$\rho_{\nu,\gamma}(\xi)=\operatorname*{esssup}_{Q\in\mathcal{Q}_\nu} E_Q\left[-\xi-\int_\nu^\gamma f\left(s,\theta_s^Q\right)ds\,\middle|\,\mathcal{F}_\nu\right],\quad P\text{-a.s.},\qquad (2)$$

where $f:[0,T]\times\Omega\times\mathbb{R}^d\to[0,\infty]$ satisfies

(f1) $f(\cdot,\cdot,z)$ is predictable $\forall\,z\in\mathbb{R}^d$;

(f2) $f(t,\omega,\cdot)$ is convex and lower semi-continuous for $dt\times dP$-a.s. $(t,\omega)\in[0,T]\times\Omega$;

(f3) $f(t,\omega,0)=0$, $dt\times dP$-a.s.; f

and $\mathcal{Q}_\nu\triangleq\left\{Q\in\mathcal{P}_\nu:\ E_Q\int_\nu^T f\left(s,\theta_s^Q\right)ds<\infty\right\}$. $R^Q(\nu)$

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Robust Optimal Stopping

In light of the robust representation (2), we can alternatively write the optimal stopping problem (1) of DCRMs as

$$\begin{aligned} &\operatorname*{esssup}_{\gamma\in\mathcal{S}_{\nu,T}}\left(\operatorname*{essinf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_\gamma+\int_\nu^\gamma f\left(s,\theta_s^Q\right)ds\middle|\mathcal{F}_\nu\right]\right) \\ &= \operatorname*{essinf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_{\tau_*(\nu)}+\int_\nu^{\tau_*(\nu)} f\left(s,\theta_s^Q\right)ds\middle|\mathcal{F}_\nu\right], \end{aligned} \tag{3}$$

$V(\nu)$

which is essentially a robust optimal stopping problem!

Some References on Robust Optimal Stopping

- Discrete-time case: (Föllmer and Schied, '04);
- Stochastic controller-stopper game: (Karatzas and Zamfirescu, '08);
- For non-linear expectation: (Bayraktar and Yao, '09).

Lower and Upper Values

To address (3), we assume

- Y is a continuous bounded process;
- $f : [0, T] \times \Omega \times \mathbb{R}^d \to [0, \infty]$ is a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}([0, \infty])$ -measurable function satisfying (f3), where $\mathscr{P}$ is the predictable σ-field on $[0, T] \times \Omega$;

(f3)

and define

- $\underline{V}(\nu) \triangleq \operatorname*{esssup}_{\gamma \in \mathcal{S}_{\nu,T}} \left(\operatorname*{essinf}_{Q \in \mathcal{Q}_\nu} E_Q \left[Y_\gamma + \int_\nu^\gamma f\left(s, \theta_s^Q\right) ds \middle| \mathcal{F}_\nu \right] \right);$ $\underline{V}(\nu)$
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as the lower and upper values at $\nu \in \mathcal{S}_{0,T}$ respectively.

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$$R^Q(\nu) \triangleq \underset{\zeta \in \mathcal{S}_{\nu,T}}{\operatorname{esssup}} E_Q\left[Y_\zeta + \int_\nu^\zeta f\left(s, \theta_s^Q\right) ds \,\middle|\, \mathcal{F}_\nu \right] \geq Y_\nu \tag{4}$$

The Classical Optimal Stopping (El Karoui, '81)

1. The process $\{R^Q(t)\}_{t \in [0,T]}$ admits an RCLL modification $R^{Q,0}$ such that $\forall \nu \in \mathcal{S}_{0,T}$, $R_\nu^{Q,0} = R^Q(\nu)$, P-a.s.

Lemma 3

2. For each $\nu \in \mathcal{S}_{0,T}$, $\tau^Q(\nu)$, the first time after ν when $R^{Q,0}$ meets Y satisfies

$$\begin{aligned} R^Q(\nu) &= E_Q\left[R^Q(\tau^Q(\nu)) + \int_\nu^{\tau^Q(\nu)} f\left(s, \theta_s^Q\right) ds \middle| \mathcal{F}_\nu \right] \\ &= E_Q\left[Y_{\tau^Q(\nu)} + \int_\nu^{\tau^Q(\nu)} f\left(s, \theta_s^Q\right) ds \middle| \mathcal{F}_\nu \right], \quad P\text{-a.s.} \end{aligned}$$

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Hence, $\tau^Q(\nu)$ is an optimal stopping time for (4).

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$\forall\, \nu \in \mathcal{S}_{0,T}$ and $k \in \mathbb{N}$, we define a subset of $\mathcal{Q}_\nu$

$$\mathcal{Q}_\nu^k \triangleq \Big\{ Q \in \mathcal{P}_\nu : \left|\theta_t^Q(\omega)\right| \vee f\left(t, \omega, \theta_t^Q(\omega)\right) \le k, \; dt \times dP\text{-a.s. on } \rrbracket \nu, T \rrbracket \Big\}.$$

Given a $Q \in \mathcal{Q}_\nu$ for some $\nu \in \mathcal{S}_{0,T}$, we *truncate* it as follows:

- $\forall\, k \in \mathbb{N}$, define a predictable set
$$A_{\nu,k}^Q \triangleq \Big\{ (t,\omega) \in \rrbracket \nu, T \rrbracket : \left|\theta_t^Q(\omega)\right| \vee f\left(t, \omega, \theta_t^Q(\omega)\right) \le k \Big\}.$$
- The predictable process $\theta^{Q^{\nu,k}} \triangleq \mathbf{1}_{A_{\nu,k}^Q} \theta^Q$ gives rise to a $Q^{\nu,k} \in \mathcal{Q}_\nu^k$ via $\frac{dQ^{\nu,k}}{dP} \triangleq \mathscr{E}\big(\theta^{Q^{\nu,k}} \bullet B\big)_T$.
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- $E_Q \int_\nu^T f\big(s,\theta_s^Q\big)ds < \infty$ and $\int_0^T \big|\theta_t^Q\big|^2 dt < \infty$, P-a.s. gives
$$\lim_{k\to\infty} \uparrow \mathbf{1}_{A_{\nu,k}^Q} = \mathbf{1}_{]\!]\nu,T]\!]}, \quad dt \times dP\text{-a.s.} \tag{5}$$

Truncation

$\forall\, \nu \in \mathcal{S}_{0,T}$ and $k \in \mathbb{N}$, we define a subset of $\mathcal{Q}_\nu$

$$\mathcal{Q}_\nu^k \triangleq \Big\{ Q \in \mathcal{P}_\nu : |\theta_t^Q(\omega)| \vee f\left(t, \omega, \theta_t^Q(\omega)\right) \le k,$$
$$dt \times dP\text{-a.s. on }]\!]\nu, T]\!] \Big\}.$$

Given a $Q \in \mathcal{Q}_\nu$ for some $\nu \in \mathcal{S}_{0,T}$, we *truncate* it as follows:

- $\forall\, k \in \mathbb{N}$, define a predictable set
$$A_{\nu,k}^Q \triangleq \Big\{ (t,\omega) \in]\!]\nu, T]\!] : |\theta_t^Q(\omega)| \vee f\left(t, \omega, \theta_t^Q(\omega)\right) \le k \Big\}.$$
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Approximating $\overline{V}$

$\forall\, \nu \in \mathcal{S}_{0,T}$, the upper value $\overline{V}(\nu)$ can be approached from above in two steps:

Lemma 1

$$\overline{V}(\nu) = \operatorname*{essinf}_{Q \in \mathcal{Q}_\nu} R^Q(\nu) = \lim_{k \to \infty} \downarrow \operatorname*{essinf}_{Q \in \mathcal{Q}_\nu^k} R^Q(\nu), \quad P\text{-a.s.}$$

Lemma 2

$\forall\, k \in \mathbb{N}$, there is a sequence $\{Q_n^{(k)}\}_{n \in \mathbb{N}} \subset \mathcal{Q}_\nu^k$ such that

Proof

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$$\operatorname*{essinf}_{Q \in \mathcal{Q}_\nu^k} R^Q(\nu) = \lim_{n \to \infty} \downarrow R^{Q_n^{(k)}}(\nu), \quad P\text{-a.s.}$$

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- $\mathcal{Q}_\nu^k \subset \mathcal{Q}_\nu^{k+1} \Longrightarrow \operatorname*{essinf}_{Q\in\mathcal{Q}_\nu} R^Q(\nu) \le \lim_{k\to\infty} \downarrow \operatorname*{essinf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu)$, P-a.s.

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- Fix $Q \in \mathcal{Q}_\nu$. We define stopping times (for *localization*):

$$\delta_m^Q \triangleq \inf\left\{t\in[\nu,T] : \int_\nu^t \left[f\left(s,\theta_s^Q\right) + |\theta_s^Q|^2\right] ds > m\right\} \wedge T$$

for any $m \in \mathbb{N}$.

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- Bayes' Rule & (f3) $\Longrightarrow E_{Q^{m,k}}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q^{m,k}})ds \middle| \mathcal{F}_\nu\right] \le$

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- D.C.T. implies that

$$E\left(\int_\nu^{\delta_m^Q}\left(\mathbf{1}_{A^Q_{\nu,k}}-1\right)\theta^Q_s dB_s\right)^2 = E\int_\nu^{\delta_m^Q}\left(1-\mathbf{1}_{A^Q_{\nu,k}}\right)\left|\theta^Q_s\right|^2 ds \xrightarrow{k\to\infty} 0$$

- Hence, up to a subsequence, $\lim_{k\to\infty} Z^{Q^{m,k}}_{\nu,T} = Z^Q_{\nu,\delta_m^Q}$, P-a.s.
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Proof of Lemma 2

- It suffices to show that $\left\{R^{Q}(\nu)\right\}_{Q\in\mathcal{Q}_\nu^k}$ is directed downwards: $\forall\, Q_1, Q_2 \in \mathcal{Q}_\nu^k$, $\exists Q_3 \in \mathcal{Q}_\nu^k$ such that

Lemma 2

$$R^{Q_3}(\nu) \le R^{Q_1}(\nu) \wedge R^{Q_2}(\nu), \quad P\text{-a.s.}$$

- Let $A \in \mathcal{F}_\nu$. $\theta_t^{Q_3} \triangleq \mathbf{1}_{\{t>\nu\}}\Big(\mathbf{1}_A\theta_t^{Q_1} + \mathbf{1}_{A^c}\theta_t^{Q_2}\Big)$ is a predictable process, thus induces a $Q_3 \in \mathcal{Q}_\nu^k$ via $\frac{dQ_3}{dP} \triangleq \mathscr{E}\left(\theta^{Q_3} \bullet B\right)_T$.
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- Let $A \in \mathcal{F}_\nu$. $\theta^{Q_3}_t \triangleq \mathbf{1}_{\{t>\nu\}}\Big(\mathbf{1}_A\theta^{Q_1}_t + \mathbf{1}_{A^c}\theta^{Q_2}_t\Big)$ is a predictable process, thus induces a $Q_3 \in \mathcal{Q}^k_\nu$ via $\frac{dQ_3}{dP} \triangleq \mathscr{E}\left(\theta^{Q_3} \bullet B\right)_T$.
- Bayes' Rule $\Longrightarrow \forall\, \gamma \in \mathcal{S}_{\nu,T}$, $E_{Q_3}\Big[Y_\gamma + \int_\nu^\gamma f\big(s, \theta^{Q_3}_s\big)ds\Big|\mathcal{F}_\nu\Big] \overset{P\text{-a.s.}}{=\!=}$

$$\mathbf{1}_A E_{Q_1}\Big[Y_\gamma + \int_\nu^\gamma f\big(s, \theta^{Q_1}_s\big)ds\Big|\mathcal{F}_\nu\Big] + \mathbf{1}_{A^c} E_{Q_2}\Big[Y_\gamma + \int_\nu^\gamma f\big(s, \theta^{Q_2}_s\big)ds\Big|\mathcal{F}_\nu\Big].$$

- Take $\underset{\gamma\in\mathcal{S}_{\nu,T}}{\mathrm{esssup}} \Longrightarrow R^{Q_3}(\nu) = \mathbf{1}_A R^{Q_1}(\nu) + \mathbf{1}_{A^c} R^{Q_2}(\nu)$, P-a.s.
- Finally, Letting $A = \{R^{Q_1}(\nu) \le R^{Q_2}(\nu)\}$ gives:

$$R^{Q_3}(\nu) = R^{Q_1}(\nu) \wedge R^{Q_2}(\nu), \quad P\text{-a.s.} \qquad \square$$

Stopping Time $\tau(\nu)$

lemma 3

$\forall\, \nu \in \mathcal{S}_{0,T}$, $\forall\, k \in \mathbb{N}$, there is a $\left\{Q_n^{(k)}\right\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu^k$ such that

$$\tau_k(\nu) \triangleq \underset{Q\in\mathcal{Q}_\nu^k}{\operatorname{essinf}}\ \tau^Q(\nu) = \lim_{n\to\infty}\downarrow \tau^{Q_n^{(k)}}(\nu), \quad P\text{-a.s.}$$

Thus $\tau_k(\nu)$ is also a stopping time in $\mathcal{S}_{\nu,T}$.

- $\mathcal{Q}_\nu^k \subset \mathcal{Q}_\nu^{k+1} \Longrightarrow \tau_k(\nu) \geq \tau_{k+1}(\nu)$. Hence

$$\tau(\nu) \triangleq \lim_{k\to\infty}\downarrow \tau_k(\nu)$$

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Properties of $\tau(\nu)$

Theorem 1

$\forall\,\nu \in \mathcal{S}_{0,T}$, we have

$$\underline{V}(\nu) = \overline{V}(\nu) = \operatorname*{essinf}_{Q \in \mathcal{Q}_\nu} E_Q\left[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f\left(s, \theta_s^Q\right) ds \,\middle|\, \mathcal{F}_\nu \right], \quad P\text{-a.s.}$$

- We shall denote the common value at ν by $V(\nu)$.
- By $\underline{V}(\nu)$, $\tau(\nu)$ is an optimal stopping time of (3).

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Pasting two Probability Measures

- Given $\nu \in \mathcal{S}_{0,T}$, let $\widetilde{Q} \in \mathcal{Q}_\nu^k$ for some $k \in \mathbb{N}$.
- $\forall\, Q \in \mathcal{Q}_\nu$, $\forall\, \gamma \in \mathcal{S}_{\nu,T}$, the predictable process

$$\theta_t^{Q'} \triangleq \mathbf{1}_{\{t \le \gamma\}} \theta_t^{Q} + \mathbf{1}_{\{t > \gamma\}} \theta_t^{\widetilde{Q}}, \quad t \in [0,T]$$

  induces a $Q' \in \mathcal{Q}_\nu$ by $\frac{dQ'}{dP} \triangleq \mathscr{E}(\theta^{Q'} \bullet B)_T$.
- $Q \in \mathcal{Q}_\nu^k \Longrightarrow Q' \in \mathcal{Q}_\nu^k$!
- Moreover, $\forall\, \sigma \in \mathcal{S}_{\gamma,T}$, $R^{Q'}(\sigma) = R^{\widetilde{Q}}(\sigma)$, P-a.s.

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In general, $\mathcal{Q}_\nu$ is *not* closed under such "pasting".

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Proof of Theorem 1

- Let $Q \in \mathcal{Q}_\nu$ and $k \le l$. $\forall\, m, n \in \mathbb{N}$, the predictable process

$$\theta_t^{Q_n^{m,k,l}} \triangleq \mathbf{1}_{\{t \le \tau_l(\nu)\}} \theta_t^{Q^{m,k}} + \mathbf{1}_{\{t > \tau_l(\nu)\}} \theta_t^{Q_n^{(l)}}, \quad t \in [0, T]$$

pastes $Q^{m,k}$ and $Q_n^{(l)}$ together to a $Q_n^{m,k,l} \in \mathcal{Q}_\nu^l$ via $\frac{dQ_n^{m,k,l}}{dP} = \mathscr{E}\left(\theta^{Q_n^{m,k,l}} \bullet B\right)_T$.

- Bayes' Rule & (f3) $\Longrightarrow \overline{V}(\nu) \le E_{Q^{m,k}}\left[\int_\nu^{\tau_l(\nu)} f(s, \theta_s^{Q^{m,k}}) ds \Big| \mathcal{F}_\nu\right]$

$$+(\|Y\|_\infty + lT) \cdot E\left[\left| Z^{Q_n^{m,k,l}}_{\nu, \tau^{Q_n^{(l)}}(\nu)} - Z^{Q^{m,k}}_{\nu, \tau_l(\nu)} \right| \Big| \mathcal{F}_\nu\right]$$

$$+E\left[Z^{Q^{m,k}}_{\nu, \tau_l(\nu)} \left(Y_{\tau^{Q_n^{(l)}}(\nu)} + k\left(\tau^{Q_n^{(l)}}(\nu) - \tau_l(\nu)\right)\right) \Big| \mathcal{F}_\nu\right]$$

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- As $l \to \infty$, D.C.T. $\Longrightarrow$

$$\overline{V}(\nu) \le E_{Q^{m,k}}\left[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f\big(s, \theta_s^{Q^{m,k}}\big)ds \Big| \mathcal{F}_\nu\right], \quad P\text{-a.s.} \quad (7)$$

- One can estimate the R.H.S. similarly to (6). Then letting $k \to \infty$ then $m \to \infty \Longrightarrow$ yields

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A more explicit optimal stopping time

- Although $\tau(\nu)$ is an optimal stopping time of (3), it is implicit.
- The "Optimal Stopping Theory" suggests that

$$\tau_V(\nu) \triangleq \inf\{t \in [\nu, T] : V(t) = Y_t\} \tag{8}$$

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Theorem 2

Let $\nu \in \mathcal{S}_{0,T}$. (1) The process $\left\{\mathbf{1}_{\{t \geq \nu\}} V\big(\tau(\nu) \wedge t\big)\right\}_{t \in [0,T]}$ admits an RCLL modification $V^{0,\nu}$ such that $\forall\, \gamma \in \mathcal{S}_{0,T}$

$$V^{0,\nu}_{\gamma} = \mathbf{1}_{\{\gamma \geq \nu\}} V\big(\tau(\nu) \wedge \gamma\big), \quad P\text{-a.s.}$$

(2) Consequently, the first time after ν when $V^{0,\nu}$ meets Y, i.e.

$$\tau_V(\nu) \stackrel{\triangle}{=} \inf\left\{t \in [\nu, T] : \; V^{0,\nu}_t = Y_t\right\}$$

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Proposition 2

Given $\nu \in \mathcal{S}_{0,T}$, $Q \in \mathcal{Q}_\nu$, and $\gamma \in \mathcal{S}_{\nu,\tau(\nu)}$, we have

$$E_Q\left[V(\gamma) + \int_\nu^\gamma f\left(s, \theta_s^Q\right) ds \,\middle|\, \mathcal{F}_\nu\right] \geq V(\nu), \quad P\text{-a.s.} \tag{9}$$

- In particular, when $Q = P$ (thus $\theta^P \equiv 0$) and $\nu = 0$,

$$\left\{V\left(t \wedge \tau(0)\right)\right\}_{t\in[0,T]}$$ is a P-submartingale!

Proof of Theorem 2 (2):

- ◂ Proposition 1 and Part (1) $\Longrightarrow V^{0,\nu}_{\tau(\nu)} = V\big(\tau(\nu)\big) = Y_{\tau(\nu)}$, P-a.s.
- Hence, $\tau_V(\nu) \le \tau(\nu)$ and $Y_{\tau_V(\nu)} = V^{0,\nu}_{\tau_V(\nu)} = V(\tau_V(\nu))$, P-a.s.
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$$\begin{aligned} V(\nu) &\le E_Q\Big[V(\tau_V(\nu)) + \int_\nu^{\tau_V(\nu)} f\big(s,\theta^Q_s\big)\,ds\Big|\mathcal{F}_\nu\Big] \\ &= E_Q\Big[Y_{\tau_V(\nu)} + \int_\nu^{\tau_V(\nu)} f\big(s,\theta^Q_s\big)\,ds\Big|\mathcal{F}_\nu\Big], \quad P\text{-a.s.} \end{aligned}$$

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- Take "$\operatorname*{essinf}_{Q\in\mathcal{Q}_\nu}$" $\Longrightarrow$

$$\begin{aligned} V(\nu) &\le \operatorname*{essinf}_{Q\in\mathcal{Q}_\nu} E_Q\Big[Y_{\tau_V(\nu)} + \int_\nu^{\tau_V(\nu)} f\big(s,\theta^Q_s\big)\,ds\Big|\mathcal{F}_\nu\Big] \\ &\le \operatorname*{esssup}_{\gamma\in\mathcal{S}_{\nu,T}} \Big(\operatorname*{essinf}_{Q\in\mathcal{Q}_\nu} E_Q\Big[Y_\gamma + \int_\nu^{\gamma} f\big(s,\theta^Q_s\big)\,ds\Big|\mathcal{F}_\nu\Big]\Big) = \underline{V}(\nu). \ \square \end{aligned}$$

Proof of Theorem 2 (2):

- Proposition 1 and Part (1) $\Longrightarrow V^{0,\nu}_{\tau(\nu)} = V\big(\tau(\nu)\big) = Y_{\tau(\nu)}$, P-a.s.
- Hence, $\tau_V(\nu) \le \tau(\nu)$ and $Y_{\tau_V(\nu)} = V^{0,\nu}_{\tau_V(\nu)} = V(\tau_V(\nu))$, P-a.s.
- Then Proposition 2 shows that $\forall\, Q \in \mathcal{Q}_\nu$

$$
\begin{aligned}
V(\nu) &\le E_Q\left[V(\tau_V(\nu)) + \int_\nu^{\tau_V(\nu)} f\big(s,\theta^Q_s\big)\,ds \Big| \mathcal{F}_\nu\right] \\
&= E_Q\left[Y_{\tau_V(\nu)} + \int_\nu^{\tau_V(\nu)} f\big(s,\theta^Q_s\big)\,ds \Big| \mathcal{F}_\nu\right], \quad P\text{-a.s.}
\end{aligned}
$$

- Take "$\underset{Q\in\mathcal{Q}_\nu}{\operatorname{essinf}}$" $\Longrightarrow$

$$
\begin{aligned}
V(\nu) &\le \underset{Q\in\mathcal{Q}_\nu}{\operatorname{essinf}}\, E_Q\left[Y_{\tau_V(\nu)} + \int_\nu^{\tau_V(\nu)} f\big(s,\theta^Q_s\big)\,ds \Big| \mathcal{F}_\nu\right] \\
&\le \underset{\gamma\in\mathcal{S}_{\nu,T}}{\operatorname{esssup}} \left(\underset{Q\in\mathcal{Q}_\nu}{\operatorname{essinf}}\, E_Q\left[Y_\gamma + \int_\nu^{\gamma} f\big(s,\theta^Q_s\big)\,ds \Big| \mathcal{F}_\nu\right] \right) = \underline{V}(\nu). \ \square
\end{aligned}
$$

An Important Observation

To determine the optimal stopping time for DCRM, knowledge of the representative function f is not necessary: If we regard the RCLL modification of $\operatorname*{esssup}_{\gamma \in \mathcal{S}_{\nu,T}} \left(-\rho_{\nu,\gamma}\left(Y_\gamma\right)\right)$, $\nu \in \mathcal{S}_{0,T}$ as the $\rho-$Snell envelope, then the first time after ν that the ρ-Snell envelope touches the reward process Y is an optimal stopping time!

The Saddle Point Problem

For any given $Q \in \mathcal{Q}_0$ and $\nu \in \mathcal{S}_{0,T}$, let us denote

$$Y_\nu^Q \triangleq Y_\nu + \int_0^\nu f(s, \theta_s^Q) ds, \quad V^Q(\nu) \triangleq V(\nu) + \int_0^\nu f(s, \theta_s^Q) ds \,.$$

Definition

A pair $(Q^*, \sigma_*) \in \mathcal{Q}_0 \times \mathcal{S}_{0,T}$ is called a saddle point for the stochastic game suggested by (3), if for every $Q \in \mathcal{Q}_0$ and $\nu \in \mathcal{S}_{0,T}$ we have

$$E_{Q^*}\left(Y_\nu^{Q^*}\right) \le E_{Q^*}\left(Y_{\sigma_*}^{Q^*}\right) \le E_Q\left(Y_{\sigma_*}^{Q}\right) . \tag{10}$$

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Sufficient Conditions for a Saddle Point

Lemma 4

A pair $(Q^*, \sigma_*) \in \mathcal{Q}_0 \times \mathcal{S}_{0,T}$ is a saddle point for the stochastic game suggested by (3), if the following conditions are satisfied:

(i) $Y_{\sigma_*} = R^{Q^*}(\sigma_*)$, P-a.s.;

(ii) for any $Q \in \mathcal{Q}_0$, we have $V(0) \leq E_Q\left[V^Q(\sigma_*)\right]$;

(iii) for any $\nu \in \mathcal{S}_{0,\sigma_*}$, we have $V^{Q^*}(\nu) = E_{Q^*}\left[V^{Q^*}(\sigma_*)\middle|\mathcal{F}_\nu\right]$, P-a.s.

Main Tools-I

Definition

We call $\mathcal{Z} \in \widehat{\mathbb{H}}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ a BMO (short for Bounded Mean Oscillation) process if

$$\|\mathcal{Z}\|_{BMO} \triangleq \sup_{\tau \in \mathcal{M}_{0,T}} \left\| E\left[\int_\tau^T |\mathcal{Z}_s|^2 ds \,\Big|\, \mathcal{F}_\tau \right]^{1/2} \right\|_\infty < \infty.$$

When $\mathcal{Z}$ is a BMO process, $\mathcal{Z} \bullet B$ is a BMO martingale; Kazamaki (1994).

Main Tools-II

Definition

BSDE with Reflection: Let $h : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\widehat{\mathscr{P}} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function. Given $S \in \mathbb{C}^0_{\mathbf{F}}[0, T]$ and $\xi \in \mathbb{L}^0(\mathcal{F}_T)$ with $\xi \geq S_T$, P-a.s., a triple $(\Gamma, \mathcal{Z}, K) \in \mathbb{C}^0_{\mathbf{F}}[0, T] \times \widehat{\mathbb{H}}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0, T]$ is called a solution to the RBSDE with terminal condition ξ, generator h, and obstacle S $\big($RBSDE (ξ, h, S) for short$\big)$, if P-a.s., we have the comparison

$$S_t \leq \Gamma_t = \xi + \int_t^T h(s, \Gamma_s, \mathcal{Z}_s)\, ds + K_T - K_t - \int_t^T \mathcal{Z}_s dB_s,$$

and the so-called flat-off condition

$$\int_0^T \mathbf{1}_{\{\Gamma_s > S_s\}} dK_s = 0, \quad P\text{-a.s.}$$

Further Assumptions on f

(H1) For every $(t,\omega) \in [0,T] \times \Omega$, the mapping $z \mapsto f(t,\omega,z)$ is continuous.

(H2) It holds $dt \times dP$-a.s. that

$$f(t,\omega,z) \geq \varepsilon \big| z - \Upsilon_t(\omega) \big|^2 - \ell , \quad \forall\, z \in \mathbb{R}^d .$$

Here $\varepsilon > 0$ is a real contant, Υ is an $\mathbb{R}^d-$valued process with $\|\Upsilon\|_\infty \triangleq \operatorname*{esssup}_{(t,\omega)\in[0,T]\times\Omega} |\Upsilon_t(\omega)| < \infty$, and $\ell \geq \varepsilon \|\Upsilon\|_\infty^2$.

(H3) The mapping $z \mapsto f(t,\omega,z) + \langle u, z \rangle$ attains its infimum over $\mathbb{R}^d$ at some $z^* = z^*(t,\omega,u) \in \mathbb{R}^d$, namely,

$$\widetilde{f}(t,\omega,u) \triangleq \inf_{z\in\mathbb{R}^d} \big(f(t,\omega,z) + \langle u,z \rangle \big) = f(t,\omega,z^*(t,\omega,u)) + \langle u, z^*(t,\omega,u) \rangle . \quad (11)$$

We further assume that there exist a non-negative BMO process ψ and a $M > 0$ such that for $dt \times dP$-a.s. $(t,\omega) \in [0,T] \times \Omega$

$$|z^*(t,\omega,u)| \leq \psi_t(\omega) + M|u|, \quad \forall\, u \in \mathbb{R}^d .$$

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Constructing a candidate saddle point

(1) Thanks to **(H2)** $\widetilde{f}$ has quadratic growth in its third argument.
(2) Thanks to Theorems 1 and 3 of Kobylanski et al. (2002), the RBSDE $(Y_T, \widetilde{f}, Y)$ admits a solution $(\widetilde{\Gamma}, \widetilde{\mathscr{Z}}, \widetilde{K}) \in \mathbb{C}^\infty_{\mathbf{F}}[0,T] \times \mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$.
(3) In fact, it can be shown that $\widetilde{\mathscr{Z}}$ is a BMO process.
(4) Since the mapping $z^* : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R}^d)$-measurable (see (H3)),

$$\theta^*_t(\omega) \stackrel{\triangle}{=} z^*(t,\omega,\widetilde{\mathscr{Z}}_t(\omega)), \quad (t,\omega) \in [0,T] \times \Omega \tag{12}$$

is a predictable process. It follows from **(H3)** that it is also a BMO process.
(5) Let $\theta^{*,\nu}_t \stackrel{\triangle}{=} \mathbf{1}_{\{t>\nu\}}\theta^*_t$, $t \in [0,T]$. Thanks to Theorem 2.3 of Kazamaki (1994) $\left\{\mathscr{E}\left(\theta^{*,\nu} \bullet B\right)_t\right\}_{t\in[0,T]}$ is a uniformly integrable martingale. Therefore, $dQ^{*,\nu} \stackrel{\triangle}{=} \mathscr{E}\left(\theta^{*,\nu} \bullet B\right)_T dP$ defines a probability measure $Q^{*,\nu} \in \mathcal{P}_\nu$.

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Is $Q^{*,\nu} \in \mathcal{Q}_\nu$?

From the Girsanov Theorem, we can deduce

$$\begin{aligned}\widetilde{\Gamma}_{\nu\vee t} &= Y_T + \int_{\nu\vee t}^{T}\left[f(s,\theta_s^{*,\nu}) + \langle \widetilde{\mathcal{Z}}_s, \theta_s^{*,\nu}\rangle\right] ds + \widetilde{K}_T - \widetilde{K}_{\nu\vee t} - \int_{\nu\vee t}^{T} \widetilde{\mathcal{Z}}_s dB_s \\ &= Y_T + \int_{\nu\vee t}^{T} f(s,\theta_s^{*,\nu}) ds + \widetilde{K}_T - \widetilde{K}_{\nu\vee t} - \int_{\nu\vee t}^{T} \widetilde{\mathcal{Z}}_s dB_s^{Q^{*,\nu}}, \qquad (13)\end{aligned}$$

where $B^{Q^{*,\nu}}$ is a Brownian Motion under $Q^{*,\nu}$. Letting $t=0$ and taking the expectation $E_{Q^{*,\nu}}$ yield that

$$E_{Q^{*,\nu}} \int_{\nu}^{T} f\left(s,\theta_s^{*,\nu}\right) ds \le E_{Q^{*,\nu}}\left(\widetilde{\Gamma}_\nu - Y_T\right) \le 2\|\widetilde{\Gamma}\|_\infty .$$

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Relating $\widetilde{\Gamma}$ to an optimal stopping problem

Lemma 5

Given $\nu \in \mathcal{S}_{0,T}$, it holds P-a.s. that

$$\widetilde{\Gamma}_t = R_t^{Q^{*,\nu},0}, \quad \forall\, t \in [\nu, T]. \tag{14}$$

Lipschitz generators

(1) Let $k \in \mathbb{N}$ and $Q \in \mathcal{Q}_\nu^k$. It is easy to see that the function $h_Q(s,\omega,z) \stackrel{\triangle}{=} f(s,\omega,\theta_s^Q(\omega)) + \langle z, \theta_s^Q(\omega)\rangle$ is Lipschitz continuous in z.

(2) Theorem 5.2 of El Karoui at al. (1997) assures that there exists a unique solution $(\Gamma^Q, \mathcal{Z}^Q, K^Q) \in \mathbb{C}_{\mathbf{F}}^2[0,T] \times \mathbb{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ to the RBSDE(Y_T, h_Q, Y). Fix $t \in [0,T]$.

(3) It holds P-a.s. that

$$\Gamma_t^Q = R_t^{Q,0}, \quad \forall\, t \in [0,T]. \tag{15}$$

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(2) Theorem 5.2 of El Karoui at al. (1997) assures that there exists a unique solution $(\Gamma^Q, \mathcal{Z}^Q, K^Q) \in \mathbb{C}^2_{\mathbf{F}}[0,T] \times \mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ to the RBSDE(Y_T, h_Q, Y). Fix $t \in [0,T]$.

(3) It holds P-a.s. that

$$\Gamma_t^Q = R_t^{Q,0}, \quad \forall\, t \in [0,T]. \tag{15}$$

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Comparison Theorem of El Karoui et al.

Proposition 3

Let $(\Gamma, \mathcal{Z}, K)$ (resp. $(\Gamma', \mathcal{Z}', K')$) $\in \mathbb{C}^2_{\mathbf{F}}[0, T] \times \widehat{\mathbb{H}}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0, T]$ be a solution of RBSDE (ξ, h, S) (resp. RBSDE (ξ', h', S')). Additionally, assume that

(i) either h or h' is Lipschitz in (y, z);

(ii) it holds P-a.s. that $\xi \le \xi'$ and $S_t \le S'_t$ for any $t \in [0, T]$;

(iii) it holds $dt \times dP$-a.s. that $h(t, \omega, y, z) \le h'(t, \omega, y, z)$ for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

Then it holds P-a.s. that $\Gamma_t \le \Gamma'_t$ for any $t \in [0, T]$.

A Key Result

From the comparison result P-a.s.

$$\widetilde{\Gamma}_t \leq \Gamma_t^Q = R_t^{Q,0}, \quad \forall\, t \in [0, T]. \tag{16}$$

Letting $t = \nu$, taking the essential infimum of the right-hand-side over $Q \in \mathcal{Q}_\nu^k$, and then letting $k \to \infty$,

$$\begin{aligned} R_\nu^{Q^{*,\nu},0} &= \widetilde{\Gamma}_\nu \leq \lim_{k\to\infty} \downarrow \operatorname*{essinf}_{Q\in\mathcal{Q}_\nu^k} R_\nu^{Q,0} = \lim_{k\to\infty} \downarrow \operatorname*{essinf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu) \\ &= \overline{V}(\nu) = V(\nu) \leq R^{Q^{*,\nu}}(\nu) = R_\nu^{Q^{*,\nu},0}, \quad P\text{-a.s.} \end{aligned}$$

As a result,

$$V(\nu) = \widetilde{\Gamma}_\nu = R_\nu^{Q^*,0} = R^{Q^*}(\nu), \quad P\text{-a.s.} \tag{17}$$

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References

(1) Optimal Stopping for Dynamic Convex Risk Measures, EB, Ioannis Karatzas and Song Yao

(2) Optimal Stopping for Nonlinear Expectations, EB and Song Yao.

Thank you for your attention.