Forward Indifference Valuation of American Options

Tim S.T. Leung

Department of Applied Math & Statistics
Johns Hopkins University

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The Merton Portfolio Optimization Problem

- On $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$, consider two liquidly traded assets: a stock $S$ & money market a.c. $B$ with +ve interest rate $(r_t)_{t \geq 0}$.
- With initial wealth $x \in \mathbb{R}$, the investor dynamically rebalances his portfolio allocations in $S$ and $B$. The discounted wealth is
  
  $$X_t^\pi = x + \int_0^t \frac{\pi_u}{S_u} dS_u,$$

  where $(\pi_t)_{t \geq 0}$ is the amount invested in $S$.
- The classical Merton portfolio optimization:
  (i) investor’s risk preference is modeled by a deterministic utility function $\hat{U}(x)$ defined at a fixed terminal time $T$;
  (ii) with wealth $X_t$, the Merton value function is
  
  $$M_t(X_t) = \text{ess sup}_{\pi \in \mathcal{Z}_{t,T}} \mathbb{E}\left\{ \hat{U}(X_T^\pi) | \mathcal{F}_t \right\}, \quad 0 \leq t \leq T.$$

- All $\hat{U}, M$, and optimal strategy $\hat{\pi}^*$ depend on $T$. 

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A Property of the Merton Value Process

- Goals: (i) specify the investor’s utility $u_0(x)$ at time 0 (not $T$); (ii) utility evolves stochastically and consistently over time.

- Observation 1: $M$ acts as the intermediate utility at time $t \leq T$.

- Observation 2: if the dynamic programming principle holds:

$$M_t(X_t) = \text{ess sup}_{\pi \in \mathcal{Z}_{t,s}} \mathbb{E} \left\{ M_s(X^\pi_s) \mid \mathcal{F}_t \right\}, \quad 0 \leq t \leq s \leq T,$$

then $(M_t(X^\pi_t))_{0 \leq t \leq T}$ is a supermartingale for any admissible strategy $\pi$, and it is a martingale under some strategy $\hat{\pi}^*$.

- With Markovian prices, the optimal portfolio allocation can be found by solving the Hamilton-Jacobi-Bellman PDE (Merton (’69) and many others).

- Exponential Utility: DPP holds in semimartingale market (Mania-Schweizer ’05); duality in terms of entropy minimization (Fritelli ’00, Delbaen et al ’02).
Definition

An $\mathcal{F}_t$-adapted process $(U_t(x))_{t \geq 0}$ is a forward performance process if:

1. $U_0(x) = u_0(x)$, for $x \in \mathbb{R}$, where $u_0 : \mathbb{R} \to \mathbb{R}$ is increasing and concave,

2. for each $t \geq 0$, $x \mapsto U_t(x)$ is increasing and concave in $x$,

3. for $0 \leq t \leq s < \infty$, we have

$$U_t(X_t) = \operatorname{ess sup}_{\pi \in \mathcal{Z}_{t,s}} \mathbb{E}\{U_s(X^\pi_s) | \mathcal{F}_t\}, \quad X_t \in \mathcal{F}_t.$$  \hfill (1)

- First introduced by Musiela-Zariphopoulou '08.
- (1) is called the horizon-unbiased cond’n in Henderson-Hobson '07, or the self-generating cond’n in Zitkovic '09.
- $(U_t(X^\pi_t))_{t \geq 0}$ is a $(\mathbb{P}, \mathcal{F}_t)$ supermartingale for any strategy $\pi$, and a martingale for some $\pi^*$ (if it exists).
An investor holds an American option with a $\mathcal{F}_t$-adapted bounded payoff process $(g_t)_{0 \leq t \leq T}$.

The holder’s value process at time $t \in [0, T]$ with wealth $X_t$ is

$$V_t(X_t) = \text{ess sup} \text{ess sup} \mathbb{E}\left\{ U_\tau(X_\tau^\pi + g_\tau) \mid \mathcal{F}_t \right\}.$$

The holder’s forward indifference price process $(p_t)_{0 \leq t \leq T}$ for the American option is defined by the equation

$$V_t(X_t) = U_t(X_t + p_t), \quad t \in [0, T].$$

Compare with the classical case:

$$\text{ess sup} \text{ess sup} \mathbb{E}\left\{ M_\tau(X_\tau^\pi + g_\tau) \mid \mathcal{F}_t \right\},$$

which corresponds to specifying that option proceeds received at exercise time $\tau$ are re-invested in the Merton portfolio up till time $T$.

In contrast, the forward performance process $U$ specifies utilities at all times, without reference to any specific horizon.
Let's model the discounted stock price as a continuous Itô process:

\[ dS_t = S_t \sigma_t (\lambda_t \, dt + \, dW_t). \]

**Theorem**

Define the stochastic process \( A_t = \int_0^t \lambda_s^2 \, ds, \ t \geq 0. \) Let the function \( u : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R} \) be \( C^{3,1} \), strictly concave and increasing in the spatial argument. Also, assume that it satisfies

\[ u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}, \]

with initial condition \( u(x, 0) = u_0(x) \), where \( u_0 \in C^3(\mathbb{R}) \). Then,

\[ U_t(x) = u(x, A_t), \quad t \geq 0, \]

defines a forward performance process. Moreover, the optimal \( \pi^* \) is

\[ \pi^*_t = -\frac{\lambda_t}{\sigma_t} \frac{u_x(X_{t^*}^\pi, A_t)}{u_{xx}(X_{t^*}^\pi, A_t)}. \]
The disc. stock price follows
\[ dS_t = \mu(Y_t)S_t \, dt + \sigma(Y_t)S_t \, dW_t. \]

The drift and volatility coefficients \( \mu(Y_t) \) and \( \sigma(Y_t) \) are driven by a non-traded stochastic factor \( Y \) which evolves according to
\[ dY_t = b(Y_t) \, dt + c(Y_t) (\rho dW_t + \sqrt{1 - \rho^2} d\hat{W}_t), \]
with correlation coefficient \( \rho \in (-1, 1) \).

Consider the exponential risk preference function \( u(x, t) \):
\[ u(x, t) = -e^{-\gamma x + \frac{t}{2}}, \]
with local risk aversion \( \gamma > 0 \).

The exponential forward performance process is given by:
\[ U^e_t(x) = -e^{-\gamma x + \frac{1}{2} \int_0^t \lambda(Y_s)^2 \, ds}, \]
where \( \lambda(y) = \mu(y)/\sigma(y) \).
The Holder’s Forward Indifference Price

- The American option has a bounded and smooth payoff function $g(s, y, t)$.
- Non-tradability of $Y$ renders the market incomplete.
- The holder’s maximal expected forward performance is

$$V_t^e(X_t) = \text{ess sup} \text{ ess sup } \mathbb{E} \left\{ -e^{-\gamma(X_\tau^{\pi} + g(S_\tau, Y_\tau, \tau))} e^{\frac{1}{2} \int_0^\tau \lambda(Y_s)^2 ds} \mid \mathcal{F}_t \right\}$$

$$= e^{\frac{1}{2} \int_0^t \lambda(Y_s)^2 ds} V(X_t, S_t, Y_t, t),$$

where

$$V(x, s, y, t) = \sup_{\tau \in T_{t,T}, \pi \in Z_{t,\tau}} \mathbb{E} \left\{ -e^{-\gamma(X_\tau^{\pi} + g(S_\tau, Y_\tau, \tau))} e^{\frac{1}{2} \int_t^\tau \lambda(Y_s)^2 ds} \mid X_t = x, S_t = s, Y_t = y \right\}.$$
The HJB Variational Inequality

We write down the associated HJB variational inequality for $V$:

\[
\begin{cases}
V_t + \mathcal{L}_{SY} V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \frac{\lambda(y)^2}{2} V \leq 0, \\
V(x, s, y, t) \geq -e^{-\gamma(x+g(s, y, t))}, \\
(V_t + \mathcal{L}_{SY} V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \frac{\lambda(y)^2}{2} V) \cdot (-e^{-\gamma(x+g)} - V) = 0, \\
V(x, s, y, T) = -e^{-\gamma(x+g(s, y, T))},
\end{cases}
\]

for $(x, s, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [0, T]$, where

\[
\mathcal{L}_{SY} v = \frac{1}{2} \sigma(y)^2 s^2 v_{ss} + \rho c(y) \sigma(y) s v_{sy} + \frac{1}{2} c(y)^2 v_{yy} + \lambda(y) \sigma(y) s v_s + b(y) v_y
\]

is the infinitesimal generator of $(S_t, Y_t)_{t \geq 0}$ under $\mathbb{P}$, and

\[
\mathcal{H}(v_{xx}, v_{xy}, v_{xs}, v_x) = \max_{\pi} \left( \frac{\pi^2 \sigma(y)^2}{2} v_{xx} + \pi \left( \rho \sigma(y) c(y) v_{xy} + \sigma(y)^2 s v_{xs} + \lambda(y) \sigma(y) v_x \right) \right).
\]
Then, the transformation $V(x, s, y, t) = -e^{-\gamma(x+p(s, y, t))}$ yields

$$
\begin{cases}
p_t + \mathcal{L}^0_{S,Y}p - \frac{1}{2}\gamma(1 - \rho^2)c(y)^2p_y^2 \leq 0, \\
p(s, y, t) \geq g(s, y, t), \\
(p_t + \mathcal{L}^0_{S,Y}p - \frac{1}{2}\gamma(1 - \rho^2)c(y)^2p_y^2) \cdot (g(s, y, t) - p(s, y, t)) = 0, \\
p(s, y, T) = g(s, y, T),
\end{cases}
$$

where $\mathcal{L}^0_{S,Y}v = \mathcal{L}_{S,Y}v - \rho c(y)\lambda(y)v_y - \lambda(y)\sigma(y)s v_s + \frac{1}{2}\sigma(y)^2 s^2 v_{ss} + \\
\rho c(y)\sigma(y)s v_{sy} + \frac{1}{2}c(y)^2 v_{yy} + (b(y) - \rho c(y)\lambda(y))v_y$.

- Note that $p(s, y, t)$ is the exponential forward indifference price and it is wealth independent.
- The optimal hedging strategy $\tilde{\pi}^*$ and exercise time $\tau_t^*$ are

$$
\tilde{\pi}^*_t = \frac{\lambda(Y_t)}{\gamma \sigma(Y_t)} + \frac{S_t}{\gamma} p_s(S_t, Y_t, t) + \frac{\rho c(Y_t)}{\gamma \sigma(Y_t)} p_y(S_t, Y_t, t),
$$

$$
\tau_t^* = \inf\{t \leq u \leq T : p(S_u, Y_u, u) = g(S_u, Y_u, u)\}.
$$
First, we define the set of equivalent local martingale measures $\mathcal{M}_f$. Define the local martingale $(Z_t^\phi)_{0 \leq t \leq T}$ by

$$Z_t^\phi = \exp\left(-\frac{1}{2} \int_0^t \lambda(Y_s)^2 + \phi_s^2 \, ds - \int_0^t \lambda(Y_s) \, dW_s - \int_0^t \phi_s \, d\hat{W}_s\right),$$

where $(\phi_t)_{0 \leq t \leq T}$ is an $\mathcal{F}_t$-adapted process such that $\mathbb{IE}^{Q^\phi}\left\{ \int_0^T \phi_t^2 \, dt \right\} < \infty$ and $\mathbb{IE}\{Z_T^\phi\} = 1$. Then, a probability measure $Q^\phi$ defined by $\frac{dQ^\phi}{d\mathbb{P}} = Z_T^\phi$ is an ELMM w.r.t. $\mathbb{P}$ on $\mathcal{F}_T$.

By Girsanov’s Theorem, $Q^\phi$, and $W_t^\phi = W_t + \int_0^t \lambda(Y_s) \, ds$ and $\hat{W}_t^\phi = \hat{W}_t + \int_0^t \phi_s \, ds$ are independent $Q^\phi$-Brownian motions.

The process $\phi$ is the risk premium for $\hat{W}$. When $\phi = 0$, we obtain the minimal martingale measure $Q^0$. 

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Treat $Q^0$ as the prior measure, and denote $Z_{t}^{\phi,0} = \mathbb{IE}^{Q^0}_t \{ \frac{dQ_{t}^{\phi}}{dQ^0} \}$.

The conditional relative entropy $H_t^{\tau}(Q^\phi|Q^0)$ of $Q^\phi$ w.r.t. $Q^0$ over the interval $[t, \tau]$ as

$$H_t^{\tau}(Q^\phi|Q^0) = \mathbb{IE}^{Q^\phi}_t \left\{ \log \frac{Z_{t}^{\phi,0}}{Z_{t}^{\phi,0}} | \mathcal{F}_t \right\} = \frac{1}{2} \mathbb{IE}^{Q^\phi}_t \left\{ \int_t^{\tau} \phi_s^2 \, ds | \mathcal{F}_t \right\}.$$

**Proposition**

The exponential forward indifference price can be represented as

$$p(S_t, Y_t, t) = \operatorname{ess} \sup_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess} \inf_{Q^\phi \in \mathcal{M}_f} \left( \mathbb{IE}^{Q^\phi}_t \{ g(S_{\tau}, Y_{\tau}, \tau) | \mathcal{F}_t \} + \frac{1}{\gamma} H_t^{\tau}(Q^\phi|Q^0) \right),$$

with the optimal risk premium $\phi_t^* = -\gamma c(Y_t) \sqrt{1 - \rho^2} p_y(S_t, Y_t, t)$.

In the classical case, the entropy term is computed w.r.t the minimal entropy martingale measure $Q^E$, instead of $Q^0$. 

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Properties of the Forward Indifference Price

The dual representation allows us to deduce the following properties:

- If $\gamma_2 \geq \gamma_1 > 0$, then $p(s, y, t; \gamma_2) \leq p(s, y, t; \gamma_1)$ and $\tau^*(\gamma_2) \leq \tau^*(\gamma_1)$ almost surely.

- As $\gamma$ increases to infinity, the penalty term vanishes, yielding

  \[
  \lim_{\gamma \to \infty} p(s, y, t; \gamma) = \sup_{\tau \in \mathcal{T}_{t,T}} \inf_{Q^\phi \in \mathcal{M}_f} \mathbb{E}^{Q^\phi} \left\{ g(S_\tau, Y_\tau, \tau) \mid S_t = s, Y_t = y \right\}.
  \]

  which is typically called the sub-hedging price (Karatzas-Kou ’98).

- As $\gamma \downarrow 0$, it is optimal not to deviate from $Q^0$ (i.e. $\phi = 0$):

  \[
  \lim_{\gamma \to 0} p(s, y, t; \gamma) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{Q^0} \left\{ g(S_\tau, Y_\tau, \tau) \mid S_t = s, Y_t = y \right\}.
  \]

- In the classical expo. utility case, the zero risk-aversion limit leads to pricing under $Q^E$ (Davis Price), not $Q^0$. 
The Classical Marginal Utility Price

The marginal utility price is the per-unit price that the investor is willing to pay for an infinitesimal position \( \delta \approx 0 \) in the claim (see Davis ’97, Kramkov-Sirbu ’06):

\[
\hat{h}_t = \frac{\mathbb{IE} \left\{ \hat{U}'(\hat{X}^*_T) C_T \mid \mathcal{F}_t \right\}}{M'_t(X_t)}, \quad t \in [0, T],
\]

where \( \hat{X}^*_T \) is the optimal Merton portfolio wealth.

We adapt this definition to the case with an American option:

\[
h_t = \sup_{\tau \in \mathcal{T}_t,T} \mathbb{IE} \left\{ M'_\tau(\hat{X}^*_\tau) g_\tau \mid \mathcal{F}_t \right\} \frac{1}{M'_t(X_t)}.
\]
Proposition

In the stochastic vol. model, consider the Merton value function

\[ M(x, y, t) = \sup_{\pi \in \mathcal{Z}_{t,T}} \mathbb{E} \left\{ \hat{U}(X_T^{\pi}) \mid X_t = x, Y_t = y \right\}. \] (2)

If \( M \) satisfies

\[ M_{xy}(x, y, t) = M_x(x, y, t) L(y, t), \] (3)

where \( L : \mathbb{R}_+ \times [0, T] \mapsto \mathbb{R} \) is a \( C^1 \) function such that the risk premium

\[ \varphi(y, t) = \sqrt{1 - \rho^2 c(y)} L(y, t), \] defines an ELMM \( Q^\varphi \).

Then, the marginal utility price for the American option \( g \) is

\[ h(s, y, t) = \sup_{\tau \in \mathcal{T}_t,T} \mathbb{E}^{Q^\varphi} \left\{ g(S_\tau, Y_\tau, \tau) \mid S_t = s, Y_t = y \right\}, \]

Note that \( h(s, y, t) \) is wealth-independent, but depends on the choice of \( \hat{U} \) (via \( L \)). When \( \hat{U}(x) = -e^{-\gamma x} \), \( Q^\varphi = Q^E \) (MEMM).
Let the discounted stock price be a continuous Itô process:

\[ dS_t = S_t \sigma_t (\lambda_t \, dt + \, dW_t) . \]

Let \( U_t(x) = u(x, A_t) \) be the investor’s forward performance process.

The marginal forward indifference price process \( (\tilde{p}_t)_{0 \leq t \leq T} \) for an American option \( g \) is defined as

\[ \tilde{p}_t = \underset{\tau \in \mathcal{T}_t, T}{\text{ess sup}} \, \mathbb{E}_{\mathcal{F}_t} \{ u_x (X_{\tau}^\pi, A_\tau) \, g_\tau | \mathcal{F}_t \} , \]

where \( A_t = \int_0^t \lambda_s^2 ds \).

As it turns out, the marginal forward indifference price is given by

\[ \tilde{p}_t = \underset{\tau \in \mathcal{T}_t, T}{\text{ess sup}} \, \mathbb{E}_{\mathcal{F}_t}^{Q^0} \{ g_\tau | \mathcal{F}_t \} , \]

where \( Q^0 \) is the minimal martingale measure \((\phi = 0)\).

Consequently (and surprisingly), \( \tilde{p}_t \) is independent of both the holder’s wealth and the choice of \( u \).
Concluding Remarks

- Forward investment performance is applicable to pricing American options.
- **Exponential** forward performance yields a dual representation that involves relative entropy minimization.
- The **MMM** $Q^0$ also acts as the pricing measure for the marginal forward indifference price, which is *wealth-independent and risk-preference independent*.

Other Applications

- Other specifications of forward performance: alternative solution to the PDE $2u_t = \left( u_x^2 / u_{xx} \right)$.
- Application to (early exercisable) ESO valuation – optimal exercise timing under forward performance.