On the dual problem associated to the robust utility maximization in a market model driven by a Lèvy Process

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Plan

- Introduction
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- The market model
- Convex measures of risk and the minimal penalty function
- Robust utility maximization

- Starting from an expected utility problem of the kind

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- Pliska provided the martingale and duality approach
  - **Pliska**,S.R. 1984 "A stochastic calculus model of continuous trading: Optimal Portfolios", Mathematics of Operations Research, 371 - 382.
- The papers
  - **Kramkov**,D. & **Schachermayer**,W. 1999 "The asymptotic elasticity of utility functions and optimal investment in incomplete markets", Ann. Appl. Probab. 9, pp. 904-950.
  - **Kramkov**,D. & **Schachermayer**,W. 2003 "Necessary and sufficient conditions in the problem of optimal investment in incomplete markets", Ann. Appl. Probab. 13,pp. 1504-1516.

The primal problem

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U \left(X_T \right) \right] \right\}. \tag{2}$$

over a set of admissible wealth processes $\mathcal{X}(x)$, lead to the dual value function

$$v_{\mathbb{Q}}(y) := \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(Y_T \right) \right] \right\}. \tag{3}$$

• **Gilboa**,I. & **Schmeidler**,D. 1989 "Maxmin expected utility with a non-unique prior", Journal of Mathematical Economics, pp. 141-153. Introduced the **"certainty-independence" axiom** what lead to robust utility functionals

$$X \longrightarrow \inf_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[U(X)]\}, \tag{4}$$

where the set of "prior" models $\mathcal{Q}$ is assumed to be a convex set of probability contents on the measurable space $(\Omega, \mathcal{F})$. The corresponding robust utility maximization problem

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[U(X)]\} \rightarrow \max, \tag{5}$$

had being considered by several authors:

Gundel,A. 2005 "Robust utility maximization for complete and incomplete market models", Finance and Stochastics 9, No. 2, pp .151-176.

- The former worst case approach do not discriminate among all the possible models in $\mathcal{Q}$, what again is reflected in inconsistencies in the axiom system proposed.
    - **Maccheroni**, **Marinacci** & **Rustichini** 2006 "Ambiguity aversion, robustness and the variational representation of preferences", Econometrica, pp. 1447 - 1498.

  introduced a relaxed axiom system which leads to utility functionals

$$X \longrightarrow \inf_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U\left(X\right)\right] + \vartheta\left(\mathbb{Q}\right)\right\}, \tag{6}$$

  where the penalty function ϑ assigns a weight $\vartheta\left(\mathbb{Q}\right)$ to each model $\mathbb{Q}\in\mathcal{Q}$.

- The corresponding dual theory for utility functions defined in the positive halfline

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[U(X_T) \right] + \vartheta(\mathbb{Q}) \right\}. \tag{7}$$

was developed in

  - **Schied**,A. 2007 "Optimal investments for risk- and ambiguity-averse preferences: a duality approach", Finance and Stochastics 11, pp. 107 - 129

introducing the robust dual value function

$$\begin{aligned} v(y) &= \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}} \left\{ v_{\mathbb{Q}}(y) + \vartheta(\mathbb{Q}) \right\} \\ &= \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}} \left\{ \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V(Y_T) \right] \right\} + \vartheta(\mathbb{Q}) \right\}. \end{aligned} \tag{8}$$

The Probability Space

- $\{L_t\}_{t\in\mathbb{R}_+}$ be a Lévy process (i.e. a cádlág process with independent stationary increments starting at zero).
- A filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with $\mathbb{F} := \left\{\mathcal{F}_t^{\mathbb{P}}(L)\right\}_{t\in\mathbb{R}_+}$ the completion of its natural filtration, i.e.

$$\mathcal{F}_t^{\mathbb{P}}(L) := \sigma\{L_s : s \leq t\} \vee \mathcal{N}$$

  where $\mathcal{N}$ is the σ-algebra generated by all $\mathbb{P}$-null sets.
- Further we denote the jump measure of L by $\mu : \Omega \times (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0)) \to \mathbb{N}$ where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$
- Recall that its dual predictable projection, also known at its Lévy system, fulfills

$$\mu^{\mathcal{P}}(dt, dx) = dt \otimes \nu(dx)$$

  where $\nu(\cdot) := \mathbb{E}[\mu([0,1] \times \cdot)]$.

- Denote the class of predictable processes $\theta \in \mathcal{P}$ integrable with respect to U^c in the sense of local martingale

$$\mathcal{L}\left(U^c\right) := \left\{\theta \in \mathcal{P} : \exists \left\{\tau_n\right\}_{n\in\mathbb{N}} \text{ sequence of stopping times with } \tau_n \uparrow \infty \text{ and } \mathbb{E}\left[\int_0^{\tau_n} \theta^2 d\left[U^c\right]\right] < \infty \ \forall n \in \mathbb{N}\right\}$$

- $\Lambda\left(U^c\right) := \left\{\int \theta_0 dU^c : \theta_0 \in \mathcal{L}\left(U^c\right)\right\}$ the linear space of processes which admits a representation as the stochastic integral w.r.t. U^c.
- We denote by $\mathcal{P} \subset \mathcal{B}\left(\mathbb{R}_+\right) \otimes \mathcal{F}$ the predictable σ-algebra and by

$$\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}\left(\mathbb{R}_0\right).$$

- The integral $\int_{\mathbb{R}_0} \theta_1 d\left(\mu - \mu^{\mathcal{P}}\right)$ is defined for processes $\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \longrightarrow \mathbb{R}$ of the class

$$\mathcal{G}\left(\mu\right) \equiv \left\{\theta_1 \in \widetilde{\mathcal{P}} : \left\{\sqrt{\int_{[0,t]\times\mathbb{R}_0} \left\{\theta_1\left(s,x\right)\right\}^2 \mu\left(ds,dx\right)}\right\}_{t\in\mathbb{R}_+} \text{ is adapted increasing loc. integ.}\right\}$$

Lemma

For any absolute continuous probability measure $\mathbb{Q} \ll \mathbb{P}$ *there are coefficients* $\theta_0 \in \mathcal{L}(W)$ *and* $\theta_1 \in \mathcal{G}(\mu)$ *such that* $\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \mathcal{E}\left(Z^\theta\right)(t)$ *for*

$$Z_t^\theta := \int_{]0,t]} \theta_0 dW + \int_{]0,t]\times\mathbb{R}_0} \theta_1(s,x)\left(\mu(ds,dx) - ds\,\nu(dx)\right). \tag{9}$$

The coefficients θ_0 *and* θ_1 *are* $\mathbb{P}$*-a.s and* $\mu_{\mathbb{P}}^{\mathcal{P}}(ds,dx)$*-a.s. unique respectively.*

Notation. We denote the class of absolute continuous probability measure w.r.t. $\mathbb{P}$ with

$$\mathcal{Q}_{\ll}(\mathbb{P})$$

and the subclass of equivalent probability measure with

$$\mathcal{Q}_{\approx}(\mathbb{P}).$$

The corresponding classes of density processes for $\mathcal{Q}_{\ll}(\mathbb{P})$ and $\mathcal{Q}_{\approx}(\mathbb{P})$ is denoted by $\mathcal{D}_{\ll}(\mathbb{P})$ and $\mathcal{D}_{\approx}(\mathbb{P})$ respectively.

The Market Model

- Let us consider an exogenous factor with a dynamic given by

$$Y_t := \int_{]0,t]} \alpha_s ds + \int_{]0,t]} \beta_s dW_s + \int_{]0,t]\times\mathbb{R}_0} \gamma(s,x)\left(\mu(ds,dx) - \nu(dx)\ ds\right),$$

where the processes α, β, γ with $\beta \in \mathcal{L}(W)$ and $\gamma \in \mathcal{G}(\mu)$ fulfill also the conditions:

$$(i) \quad \int_{]0,t]} (\alpha_s)^2 ds < \infty \quad \forall t.$$

$$(ii) \quad \gamma \geq -1\ \mathbb{P}-a.s. \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0$$

$$(iii) \quad \gamma \text{ is a locally bounded process}$$

- The process Y specifies the discounted price process as its Doleans-Dade exponential

$$S_t = S_0\mathcal{E}(Y_t) = S(0) + \int_0^t S_{u-} dY_u,$$

- Further let the predictable cadlag process $\{\pi_t\}_{t\in\mathbb{R}_+}$ with $\int_0^t (\pi_s)^2 \, ds < \infty$ $\mathbb{P}$-a.s. $\forall t \in \mathbb{R}_+$ denotes the proportion of the wealth at time t invested in the risky asset S at this time. For an initial capital x the discounted wealth $X_t^{x,\pi}$ associated to a self-financing admissible investment strategy π fulfills the equation

$$X_t^{x,\pi} = x + \int_0^t \frac{X_{u-}^{x,\pi}\pi_u}{S_{u-}} dS_u.$$

- An strategy $\{\pi_t\}_{t\in\mathbb{R}_+}$ with initial capital x is called admissible when the wealth process $X_t^{x,\pi} \geq 0$ $\forall t$ and the class of such wealth processes is denoted by $\mathcal{X}(x)$.

- Our next result characterizes the class of equivalent local martingale measures

$$\mathcal{Q}_{elmm}(\mathbb{P}) := \{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : \mathcal{X}(1) \subset \mathcal{M}_{loc}(\mathbb{Q})\}.$$

Theorem

Given $\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P})$ let $\theta_0 \in \mathcal{L}(W)$, $\theta_1 \in \mathcal{G}(\mu)$ be the corresponding processes obtained in Lemma 1. Then the following equivalence holds:

$$\mathbb{Q} \in \mathcal{Q}_{elmm}(\mathbb{P}) \iff \alpha_t + \beta_t \theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,x)\,\theta_1(t,x)\,\nu(dx) = 0 \; \forall t \geq 0$$

- Denote by $\mathcal{Q}_{cont}(\Omega, \mathcal{F})$ the set of **probability contents** on the measurable space $(\Omega, \mathcal{F})$ (i.e. finite additive set functions $\mathbb{Q} : \mathcal{F} \to [0, 1]$ with $\mathbb{Q}(\Omega) = 1$)
- Let $\mathcal{Q}(\Omega, \mathcal{F}) \subset \mathcal{Q}_{cont}(\Omega, \mathcal{F})$ be the family of probability measures.
- From the general theory of convex risk measures, we know that any functional

$$\psi : \mathcal{Q}_{cont}(\Omega, \mathcal{F}) \to \mathbb{R} \cup \{+\infty\}$$

with

$$\inf_{\mathbb{Q} \in \mathcal{Q}_{cont}} \psi(\mathbb{Q}) > -\infty$$

induce a convex measure of risk as an application

$$\rho : \mathfrak{M}_b(\Omega, \mathcal{F}) \to \mathbb{R}$$

given by

$$\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{cont}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \psi(\mathbb{Q})\}. \tag{10}$$

- Let now h_0 and h_1 be $\mathbb{R}_+$-valued convex, lower semicontinuous functions with $h_0(0) = 0 = h_1(0)$ which satisfy the conditions

$$
\begin{aligned}
h_0(x) &\geq \kappa_1 x^2 - \kappa_2, \\
h_1(x) &\geq 2\kappa_1 x \ln(1+x) \vee |x| \vee |(1+x)\ln(1+x)|,
\end{aligned}
$$

for some constants $\kappa_1, \kappa_2 > 0$. Further define the penalty function

$$
\begin{aligned}
\vartheta(\mathbb{Q}) &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T h_0(\theta_0(t))\,dt + \int_{[0,T]\times\mathbb{R}_0} h_1(\theta_1(t,x))\,\mu_{\mathbb{P}}^{\mathcal{P}}(dt,dx)\right]\mathbf{1}_{\mathcal{Q}_{\ll}} \\
&\quad +\infty \times \mathbf{1}_{\mathcal{Q}_{cont}\setminus\mathcal{Q}_{\ll}}(\mathbb{Q}),
\end{aligned}
$$

where θ_0, θ_1 are the processes associated to $\mathbb{Q}$ from Lemma 1, and the convex measure of risk

$$
\rho(X) := \sup_{\mathbb{Q}\in\mathcal{Q}_{\ll}(\mathbb{P})} \left\{\mathbb{E}_{\mathbb{Q}}[-X] - \vartheta(\mathbb{Q})\right\}. \tag{12}
$$

• Any convex measure of risk ρ on the space of bounded measurable functions $\mathfrak{M}_b(\Omega, \mathcal{F})$ is of the form

$$\rho(X) := \sup_{\mathbb{Q}\in\mathcal{Q}_{cont}} \left\{ \mathbb{E}_{\mathbb{Q}}[-X] - \psi_\rho^*(\mathbb{Q}) \right\},$$

where

$$\psi_\rho^*(\mathbb{Q}) = \sup_{X\in\mathcal{A}\rho} \mathbb{E}_{\mathbb{Q}}[-X]$$

and $\mathcal{A}_\rho := \{X \in \mathfrak{M}_b : \rho(X) \leq 0\}$ is the acceptance set of ρ. $\psi_\rho^*(\mathbb{Q})$ is called the **minimal penalty function** asociated to ρ and fulfills the biduality relation

$$\psi_\rho^*(\mathbb{Q}) = \sup_{X\in\mathfrak{M}_b(\Omega,\mathcal{F})} \{\mathbb{E}_{\mathbb{Q}}[-X] - \rho(X)\} \quad \forall\mathbb{Q} \in\mathcal{Q}_{cont}. \qquad (13)$$

Theorem

Let $\psi : \mathcal{Q}_{\ll}(\mathbb{P}) \rightarrow \mathbb{R} \cup \{+\infty\}$ *be a function with* $\inf_{\mathbb{Q} \in \mathcal{Q}_{cont}} \psi(\mathbb{Q}) > -\infty$ *and* $\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[-X] - \psi(\mathbb{Q})\}$ *the associated convex measure of risk. The penalty* ψ *is the minimal penalty function asociated to* ρ *i.e.* $\psi = \psi_{\rho}^{*}$ *if* ψ *is a proper convex function and lower semicontinuous w.r.t. the weak topology* $\sigma\left(L^1, L^\infty\right)$.

Theorem

The penalty function ϑ *as defined in* (11) *is the minimal penalty function of the convex risk measure* ρ *given by* (12).

- $U : (0, \infty) \longrightarrow \mathbb{R}$ is strictly increasing, strictly concave, continuous differentiable, which satisfies the Inada conditions (i.e. $U'(0+) = +\infty$ and $U'(\infty-) = 0$) with asymptotic elasticity strictly less than one.
- Let us now introduce the class

$$\mathcal{C} := \left\{ \mathcal{E}\left(Z^{\xi}\right) : \begin{array}{l} \xi := \left(\xi^{(0)}, \xi^{(1)}\right), \ \xi^{(0)} \in \mathcal{L}(W), \ \xi^{(1)} \in \mathcal{G}(\mu), \text{ with} \\ \alpha_t + \beta_t \xi_t^{(0)} + \int_{\mathbb{R}_0} \gamma(t,x)\, \xi^{(1)}(t,x)\, \nu(dx) = 0, \ \forall t \end{array} \right.$$

with Z^{ξ} defined as in (9), and observe that

$$\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1),$$

where

$$\mathcal{Y}_{\mathbb{Q}}(y) := \{ Y \geq 0 : \mathbb{Q}\text{-supermartingale}, \ Y_0 = y, \ YX \ \mathbb{Q}\text{-supermartingale}$$

- If

$$v_{\mathbb{Q}}(y) < \infty \quad \forall \mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta} \quad \forall y > 0. \tag{14}$$

we have from Theorem 2 in [Krk&Scha 2003] that

$$u_{\mathbb{Q}}(x) < \infty \quad \forall \mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta} \quad \forall x > 0 \tag{15}$$

Theorem

For an utility function U, which fulfills the condition (14), we have that the dual value function turn into

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}} \left\{ \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\} \tag{16}$$

Lemma

For $U(x) = \log(x)$ we have (14).