Portfolio insurance under risk-measure constraint

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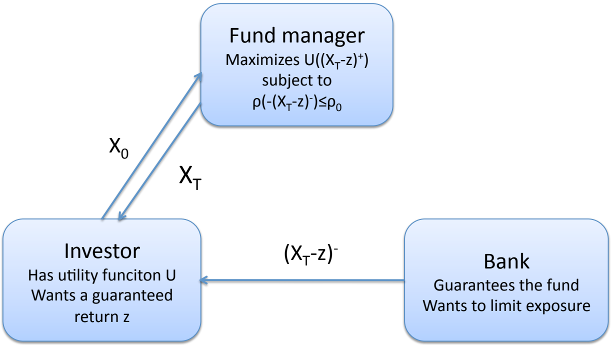
The Insurance



Market assumptions

We will assume that:

- The market is complete with a unique martingale measure $\xi\mathbb{P}$ on $(\Omega, \mathscr{F})$
- The risk is measured in terms of a law-invariant convex risk measure ρ continuous from above.

$$\rho(X) := \sup_{Q \in \mathscr{M}_1(\mathbb{P})} \left(\mathbb{E}_Q[-X] - \gamma_{min}(Q) \right)$$

  we will suppose $\rho(0) = 0$
- The risk exposure imposed on the Fund manager is given by ρ_0

Setting

If we let

$$H := \left\{ X \in \mathbb{L}^1(\mathbb{P}) \middle| \mathbb{E}[\xi X] \leq x_0, 0 \leq \rho\left(-(X-z)^-\right) \leq \rho_0 \right\}$$

then the **FM**'s aim is to find, if it exists, a $X^* \in H$ such that:

$$\mathbb{E}\left[u(X^*-z)^+\right] = \sup_{X \in H} \mathbb{E}\left[u(X-z)^+\right]$$

and the optimal payoff for the Investor will be

$$\max(X^*, z)$$

Decoupling-Idea

Define $U(X) := \mathbb{E}\left[u\left((X-z)^{+}\right)\right]$ and remark that

$$U(X) = U(X\mathbf{1}_A)$$

where $A := \{X \geq z\}$. This means that only $X\mathbf{1}_A$ remains important for the investor. This remark suggests this decoupling:

Decoupling-Idea

let $(A, x^+) \in \mathscr{F} \times \mathbb{R}^+$ and

$$\mathscr{P}_1 : \begin{cases} \sup U(X) & \text{s.t.} \\ \mathbb{E}[\xi X] \leq x^+, & X \in \mathbb{L}^1(\mathbb{P}) \text{ and} \\ X = 0 \quad \text{on } A^c, & X \geq z \quad \text{on } A \end{cases}$$

and

$$\triangle(A) : \begin{cases} \inf \mathbb{E}[\xi Y] & \text{s.t.} \\ \rho\left(-(Y - z)^- \mathbf{1}_{A^c}\right) \leq \rho_0, & Y \in \mathbb{L}^1(\mathbb{P}) \\ Y = 0 \quad \text{on } A, & Y \leq z \quad \text{on } A^c \end{cases}$$

Define also $x_+(A) := x_0 - \triangle(A)$. Remark upon how both these problems can be solved by Lagrangian methods.

Decoupling-Idea

The next example will clarify the role of $\triangle(A)$. Fix A such that $0 < \mathbb{P}(A) < 1$ and suppose $\triangle(A) = -\infty$. It is possible to find , $\forall n \in \mathbb{N}$ a $Y^n \in \mathscr{P}_2(A)$ such that $\mathbb{E}[\xi Y^n] \leq -n$. Consider now this payoff

$$X^n = \frac{x_0 + n}{\mathbb{E}[\xi \mathbf{1}_A]} \mathbf{1}_A + Y^n$$

We deduce $X^n \in H$ and $U(X^n) \to +\infty$, which means that our problem has no finite solution.
We will then carry out the following:

Assumption

$$\inf_{A \in \mathscr{F}} \triangle(A) > -\infty$$

Decoupling-Idea

The following condition guarantees our assumption:

Theorem

Let ρ be a law-invariant convex risk measure and ξ the risk-neutral probability of the market.
If

$$\gamma_{min}(\xi\mathbb{P}) < +\infty$$

then $\inf_A \triangle(A) > -\infty$.

Decoupling

Let $X\left(A, x^{+}\right)$ the solution of problem $\mathscr{P}_1$ with parameters A and x^{+} and recall that $x^{+}(A) := x_0 - \triangle(A)$

Theorem

If $\inf_A \triangle(A) > -\infty$ *then*

$$\sup_{X \in H} U(X) = \sup_{A \in \mathscr{F}} U\left(X\left(A, x^{+}(A)\right)\right)$$

If $\inf_A \triangle(A) = -\infty$ *then we already know*

$$\sup_{X \in H} U(X) = +\infty$$

Algorithm 1

Using the last Theorem, we can solve our problem as the following:

1. fix $A \in \mathscr{F}$
2. solve $\mathscr{P}_2(A)$ and find $\triangle(A)$
3. solve $\mathscr{P}_1(A)$ with parameter $x^+(A)$
4. maximize the value function of $\mathscr{P}_1(A)$, $U(X(A, x^+(A)))$, over $A \in \mathscr{F}$

Decoupling

We can use the last result to give a necessary and sufficient condition for the existence of a finite solution

Theorem

Assume $\inf_A \triangle(A) > -\infty$ *and* X^* *is optimal for our problem. Define* $A^* := \{X^* \geq z\}$. *One has*

$$\sup_{A \in \mathscr{F}} U\left(X\left(A, x^+(A)\right)\right) = U\left(X\left(A^*, x^+(A^*)\right)\right)$$

$$\triangle(A^*) = \mathbb{E}\left[\xi Y^*\right], \textit{ where } Y^* := X^* - X^* \mathbf{1}_{A^*}$$

Decoupling

Reciprocally, let $A^* \in \mathscr{F}$ and a $Y^* \in \mathscr{P}_2(A^*)$ such that

$$U\left(X\left(A^*, x^+(A^*)\right)\right) = \sup_{A \in \mathscr{F}} U\left(X\left(A, x^+(A)\right)\right)$$

$$\mathbb{E}[\xi Y^*] = \triangle(A^*) = \inf_{Y \in \mathscr{H}_2(A^*)} \mathbb{E}[\xi Y]$$

Then a solution of our problem is given by

$$X^* := X\left(A^*, x^+(A^*)\right) \mathbf{1}_{A^*} + Y^* \mathbf{1}_{A^{*,c}}$$

In this case, the payoff for the investor will be

$$Payoff = X\left(A^*, x^+(A^*)\right) \mathbf{1}_{A^*} + z$$

A $\mathbb{R}$-valued Maximization Problem

- Generally a maximization over the sets in $\mathscr{F}$ is not simple
- Our aim here is to show that this latter maximization may be carried out over a subset of $\mathcal{F}$, parameterized by a real number, Jin and Zhou (2008).

define

$$v(A) := \sup_{X \in \mathscr{P}_1(A, x^+(A))} U(X)$$

so then

$$\sup_{X \in H} U(X) = \sup_{A \in \mathscr{F}} U\left(X\left(A, x^+(A)\right)\right) = \sup_{A \in \mathscr{F}} v(A)$$

A $\mathbb{R}$-valued Maximization Problem

Theorem

Suppose ξ has not atoms. Define $\underline{\xi} := \text{essinf}\,\xi$ and $\overline{\xi} := \text{esssup}\,\xi$. Let $A \in \mathscr{F}$ and $c \in \left[\underline{\xi}, \overline{\xi}\right]$ such that $\mathbb{P}(\xi \leq c) = \mathbb{P}(A)$. Then

$$v(A) \leq v(\{\xi \leq c\})$$

which means

$$\sup_{X \in H} U(X) = \sup_{A \in \mathscr{F}} v(A) = \sup_{c \in \left[\underline{\xi}, \overline{\xi}\right]} v(\{\xi \leq c\})$$

Algorithm 2

Using the last Theorem we can solve our problem as the following:

1. fix $c \in [\underline{\xi}, \overline{\xi}]$
2. solve $\mathscr{P}_2(c)$ and find $\triangle(c)$
3. solve $\mathscr{P}_1(c)$ with parameter $x_+(c) = x_0 - \triangle(c)$
4. find c^* that maximizes $U(X_1(\{\xi \leq c\}, x_+(c)))$
5. A optimal payoff for the Investor will be $X^* = X_1(\{\xi \leq c\}, x_+(c)) \mathbf{1}_{\{\xi \leq c\}} + z$

Example-CVaR

We will now see what happens when $\rho = CVaR_\lambda$, $\lambda \in (0,1)$:

$$CVaR_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda VAR_u(X)\,du$$

or, equivalently

$$CVaR_\lambda(X) = \int_0^{+\infty} \psi_\lambda(\mathbb{P}(-X > t))\,dt$$

$$\text{where } \psi_\lambda(u) = \frac{(u \wedge \lambda)}{\lambda}$$

Example-CVaR

We then have the following:

Theorem

Let ξ the state price density.

i) *If ξ is unbounded then our problem has no finite solution*

ii) *If ξ is bounded then our value function is:*

$$\sup_{X \in H} U(X) = \sup_{c \in [\underline{\xi}, \overline{\xi}]} \mathbb{E}\left[u\left(\left[I\left(\lambda(c)\,\xi\right)\right]^{+}\right) \mathbf{1}_{\{\xi \leq c\}}\right]$$

Example-CVaR

where

- $I = (u')^{-1}$
- $\lambda(c)$ *is given by:* $\mathbb{E}\left[\xi\left([I(\lambda(c)\xi)]^{+}\right)\mathbf{1}_{\{\xi \leq c\}}\right] = x_0 + \rho_0 \beta \bar{\xi}$

We do not have a solution for the Fund Manager problem because problem $\mathscr{P}_2$ does not have a minimum. However we can give a solution for the investor which is

$$X^* = z + [I(\lambda(c^*)\xi)]^{+}$$

Example-CVaR

Note also that the minimal penalty function for the $CVaR_\lambda$ is given by:

$$\gamma_{min}(Q) := \begin{cases} 0 & \text{if } \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\lambda}, \quad \mathbb{P}\text{-a.s} \\ +\infty & \text{otherwise} \end{cases}$$

So, for example, if we have ξ bounded but $\mathbb{P}\left(\xi > \frac{1}{\lambda}\right) > 0$ then it turns out $\gamma_{min}(\xi\mathbb{P}) = +\infty$ even if the problem has a solution! Here is a good example where we have a solution even if $\gamma_{min}(\xi\mathbb{P}) = +\infty$!

Example-Entropic Risk Measure

If we consider $\rho = ERM_\lambda$, where $\lambda > 0$ and

$$ERM_\lambda(X) := \lambda \ln \mathbb{E}\left[\exp\left(-\frac{1}{\lambda}X\right)\right]$$

We have:

Example-Entropic Risk Measure

Theorem

Assume that the state price density ξ *has no atoms and satisfies* $\xi \log \xi \in \mathbb{L}^1(\mathbb{P})$. *Then the optimal payoff for the fund manager is given by*

$$X^* := z + [I(\lambda(c^*)\xi)]^+ \mathbf{1}_{\{\xi \leq c^*\}} - \beta \left[\log\left(\frac{\beta}{\eta(c^*)}\xi\right)\right]^+ \mathbf{1}_{\{\xi > c^*\}}$$

Example-Entropic Risk Measure

where

- $I = (u')^{-1}$
- $\lambda(c)$ *is given by:* $\mathbb{E}\left[\xi \left[I\left(\lambda(c)\,\xi\right)\right]^{+} \mathbf{1}_{\{\xi \le c\}}\right] = x_0 - \triangle(c)$
- $\alpha(c) = \mathbb{P}(\xi > c)$
- $\psi(c) := \mathbb{E}\left[\xi \mathbf{1}_{\{\xi > c\}}\right]$
- $\triangle(c) = -\beta\left(\log\left(\frac{\beta}{\eta(c)}\right)\psi\left(c \vee \frac{\eta(c)}{\beta}\right) + \hat{\psi}\left(c \vee \frac{\eta(c)}{\beta}\right)\right)$
- $\eta(c)$ *is given by:*
  $\frac{\beta}{\eta(c)}\psi\left(c \vee \frac{\eta(c)}{\beta}\right) + \mathbb{P}\left(c < \xi \le \frac{\eta(c)}{\beta}\right) = e^{\frac{\rho_0}{\beta}} + \alpha(c) - 1$
- c^* *attains the supremum of* $c \to \mathbb{E}\left[u\left(\left[I\left(\lambda(c)\,\xi\right)\right]^{+}\right)\mathbf{1}_{\{\xi \le c\}}\right]$

Example-Entropic Risk Measure

Again, the proof is not complicated; one just needs to follow the **Algorithm 2**.
Remark that the penalty function for the ERM_λ:

$$\gamma_{min}(Q) := \lambda H(Q\,|\mathbb{P}) := \lambda \mathbb{E}_Q\left[\log\left(\frac{dQ}{d\mathbb{P}}\right)\right]$$

With our hypothesis, we easily have $\gamma_{min}(\xi\mathbb{P}) < \infty$: we know that this is a sufficient condition under which the problem has a solution. The condition $\xi \log \xi \in \mathbb{L}^1(\mathbb{P})$ is naturally verified in a Black-Scholes framework.

Numerical Results

We will see now what happens in a very simple one-dimensional Black-Scholes model:
On $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, let

$$dS_t = S_t \left(bdt + \sigma dW_t\right) \; S_0 = 1$$

and suppose $\mu = b/\sigma > 0$. The unique equivalent martingale measure is given by $\mathbb{Q} = \xi\mathbb{P}$, where
$\xi = \exp(-\mu W_T - \mu^2 T/2) = \left[S_T \exp\left(T\left(\sigma^2 - b\right)/2\right)/S_0\right]^{-\frac{b}{\sigma^2}}$.

Numerical Results

We will use the utility function $u(x) = 1 - e^{-\delta x}$ and the ERM_λ as risk measure. Our initial data is:

Data	
drift	15%
volatility	40%
risk premium	1.5
maturity	1 **year**
initial capital	10
guarantee	8.5
risk tolerance	1.5
entropic constant (λ)	0.5
utility constant (δ)	0.6

Numerical Results

An optimal payoff will be:

$$X^* := \left[\frac{L}{\delta}\log(S_T) + K_1\right]^+ \mathbf{1}_{\{S_T \geq s^*\}} - \beta\left[K_2 - L\log(S_T)\right]^+ \mathbf{1}_{\{S_T < s^*\}} + z$$

where

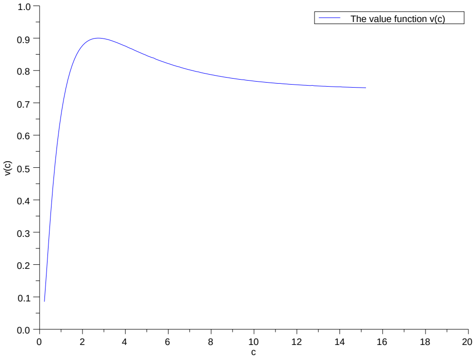
$$s^* = 0.9375, \quad K_1 = 1.34026, \quad K_2 = 3.18886$$

Other quantities one can also compute are optimal c^*, value functions of problems $\mathscr{P}_1$-$\mathscr{P}_2$ and the "success" probability:

$$c^* = 2.72293, \quad v(c^*) = 0.900134$$
$$\triangle(c^*) = -1.17387, \quad \mathbb{P}(S_T \geq s^*) = 0.946722$$

Graphics

The following figure is the value function $c \to v(c)$:



Graphics

The Payoff profile for the Fund Manager

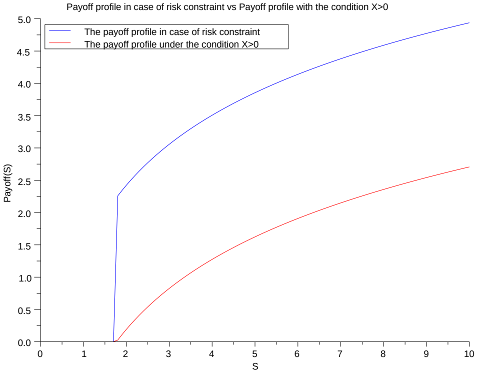
Graphics

Suppose, for sake of simplicity, $z = 0$ and let us see what happens if we do not allow any risk, i.e. $\rho_0 = 0$. We can see this by solving the following problem

$$\max \mathbb{E}[1 - \mathrm{e}^{-\delta X^+}]$$
$$\mathbb{E}[X] \leq x_0, \quad X \geq 0$$

and compare the payoff profiles

Graphics


Payoff profile in case of risk constraint vs Payoff profile with the condition X>0
The payoff profile in case of risk constraint
The payoff profile under the condition X>0
Payoff(S)
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
6
7
8
9
10
S