Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles

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http://arxiv.org/abs/1002.3627
(joint work with Hans Föllmer and Irina Penner)

Static risk measures on random variables

Origin: axiomatic analysis of capital requirements needed to cover the risk of future liabilities.

- A **static risk measure** is a map $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ satisfying certain axioms
- L^∞ - set of discounted terminal values of financial positions
- $\rho(X)$ - minimal amount of cash that has to be added to the financial position X in order to make it acceptable

(Artzner, Delbaen, Eber &Heath(1997,99), Föllmer&Schied(2002), Frittelli & Rosazza Gianin(2002),...)

Conditional risk measures on processes

In the static setting: the role of information is not visible and the timing of payments not considered.

- A **conditional risk measure on processes** is a map $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ satisfying analogous axioms
- $\mathcal{R}_t^\infty$: bounded adapted processes from time t on
  - set of cumulated cash flows (value processes)
- $\rho_t(X)$ - minimal conditional capital that has to be added to the cash flow X at time t in order to make it acceptable

(Cheridito, Delbaen & Kupper (2004,05,06), Artzner, Delbaen, Eber, Heath & Ku (2007),...)

Dynamical setting

Discrete-time setting, with finite or infinite time horizon T:

- $T \in \mathbb{N}$, time axis $\mathbb{T} = \{0, 1, ..., T\}$
- $T = \infty$, time axis $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$

Multiperiod information structure: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$

$\mathcal{R}^\infty$ = bounded adapted processes on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$
= (cumulated) cash flows

$\mathcal{R}^\infty_t$ = cash flows from time t on

Conditional convex risk measures

$\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ is called a **conditional convex risk measure** for processes if for all $X, Y \in \mathcal{R}_t^\infty$:

- Normalization: $\rho_t(0) = 0$
- Monotonicity: $X \le Y \;\Rightarrow\; \rho_t(X) \ge \rho_t(Y)$
- Conditional convexity: $\forall \lambda \in L^\infty(\Omega, \mathcal{F}_t, P)$, $0 \le \lambda \le 1$:
  $\rho_t(\lambda X + (1-\lambda)Y) \le \lambda \rho_t(X) + (1-\lambda)\rho_t(Y)$
- Conditional cash-invariance:
  $\rho_t(X + m 1_{\{t,t+1,\dots\}}) = \rho_t(X) - m, \quad m \in L^\infty(\Omega, \mathcal{F}_t, P)$

▶ The timing of the payment is taken into account

$(\rho_t)_t$ is called **dynamic convex risk measure** for processes

Product space and optional filtration

- Define the **product space** $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ as: $\bar{\Omega} = \Omega \times \mathbb{T}$,

$$\bar{\mathcal{F}} = \sigma(\{A_t \times \{t\} \mid A_t \in \mathcal{F}_t,\ t \in \mathbb{T}), \quad \bar{P} = P \otimes \mu,$$

where $\mu = (\mu_t)_{t \in \mathbb{T}}$ is some adapted reference process s.t. $\mu_t > 0$ and $\sum_t \mu_t = 1$, and $E_{\bar{P}}[X] := E_P\left[\sum_t X_t \mu_t\right]$

- Consider the **optional filtration** $(\bar{\mathcal{F}}_t)_{t \in \mathbb{T}}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$, given by

$$\bar{\mathcal{F}}_t = \sigma\left(\{A_j \times \{j\}, A_t \times \{t, ..\} | A_j \in \mathcal{F}_j, j = 0, .., t-1, A_t \in \mathcal{F}_t\}\right)$$

$$\Longrightarrow \qquad \boxed{\mathcal{R}^\infty = L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})}$$

Risk measures viewed on the optional filtration

Theorem. There is a **one-to-one correspondence** between

- conditional convex risk measures for processes

$$\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$$

- conditional convex risk measures for random variables **on the product space**

$$\bar{\rho}_t : L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) \to L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P})$$

The relation is given by

$$\bar{\rho}_t(X) = -X_0 1_{\{0\}} - \ldots - X_{t-1} 1_{\{t-1\}} + \rho_t(X) 1_{\{t,t+1,\ldots\}}$$

Representation of risk measures on random variables

Theorem. For $\rho_t : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ TFAE:

1. ρ_t is continuous from above: $X^n \searrow X \Rightarrow \rho_t(X^n) \nearrow \rho_t(X)$
2. ρ_t has the following **robust representation**:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X|\mathcal{F}_t] - \alpha_t(Q))$$

where

$$\mathcal{Q}_t = \left\{ Q \ll P \;\middle|\; Q = P|_{\mathcal{F}_t} \right\},$$

and the **minimal penalty function** α_t is given by

$$\alpha_t(Q) = \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F})} (E_Q[-X|\mathcal{F}_t] - \rho_t(X))$$

(Detlefsen and Scandolo (2005))

Optional random measures

For a measure $Q \ll_{loc} P$ we introduce:

- the set $\Gamma(Q)$ of **optional random measures** γ on $\mathbb{T}$ which are normalized with respect to Q:

$\gamma = (\gamma_t)_{t\in\mathbb{T}}$ nonnegative adapted process s.t. $\sum_{t\in\mathbb{T}} \gamma_t = 1$ Q-a.s.

with the additional property

$$\gamma_\infty = 0 \;\; Q\text{-a.s. on } \left\{ \lim_{t\to\infty} \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \infty \right\} \text{ if } \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$$

Predictable discounting processes

For a measure $Q \ll_{loc} P$ we introduce:

- the set $\mathcal{D}(Q)$ of **predictable discounting processes** D:

$$D = (D_t)_{t \in \mathbb{T}} \text{ predict. non-increasing, } D_0 = 1,\ D_\infty = \lim_{t \to \infty} D_t \quad Q\text{-a.s.}$$

where

$$D_\infty = 0 \quad Q\text{-a.s.} \quad \text{if} \quad \mathbb{T} = \mathbb{N}_0,$$

$$D_\infty = 0 \quad Q\text{-a.s. on } \left\{ \lim_{t \to \infty} \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \infty \right\} \quad \text{if } \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$$

►► There is a **one-to-one correspondence** between optional random measures in $\Gamma(Q)$ and predictable discounting in $\mathcal{D}(Q)$:

$$\gamma_t = D_t - D_{t+1}, t < \infty, \quad \gamma_\infty = D_\infty$$

Decomposition of measures on the optional σ-field

Theorem. For any probability measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ we have: $\bar{Q} \ll \bar{P}$ if and only if there exist

- a probability measure Q on $(\Omega, \mathcal{F}_T)$, $Q \ll_{loc} P$
- an optional random measure $\gamma \in \Gamma(Q)$ (resp. $D \in \mathcal{D}(Q)$)

such that

$$E_{\bar{Q}}[X] = E_Q\left[\sum_{t\in\mathbb{T}} \gamma_t X_t\right] = E_Q\left[\sum_{t=0}^{T} D_t \Delta X_t\right], \quad X \in \mathcal{R}^\infty$$

(combining the Itô-Watanabe factorization with an extension theorem for standard systems)

In this case we write: $\bar{Q} = Q \otimes \gamma = Q \otimes D$

Robust representation

Theorem. For $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ TFAE:

1. ρ_t continuous from above: $X_s^n \searrow X_s \, \forall \, s \geq t \Rightarrow \rho_t(X^n) \nearrow \rho_t(X)$
2. ρ_t has the following **robust representation**:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{\text{loc}}} \operatorname*{ess\,sup}_{D \in \mathcal{D}_t(Q)} \left(E_Q\left[-\sum_{s=t}^{T} D_s \Delta X_s \,\Big|\, \mathcal{F}_t \right] - \alpha_t(Q \otimes D) \right),$$

↗ model ambiguity

↖ discounting ambiguity

$$\mathcal{Q}_t^{\text{loc}} = \{ Q \ll_{loc} P : Q = P|_{\mathcal{F}_t} \}, \; \mathcal{D}_t(Q) = \{ D \in \mathcal{D}(Q) : D_s = 1 \, s \leq t \}$$

$$\alpha_t(Q \otimes D) = \operatorname*{Q\text{-}ess\,sup}_{X \in \mathcal{R}_t^\infty} \left(E_Q\left[-\sum_{s \geq t} \frac{\gamma_s}{D_t} X_s \,\Big|\, \mathcal{F}_t \right] - \rho_t(X) \right)$$

Time consistency

$X \in \mathcal{R}^\infty \;\to\; (\rho_t(X))_t$ describes the evolution of risk over time.

Question: How should **risk measurement** be **updated** as more information becomes available?

- $(\rho_t)_t$ is called (strongly) time consistent if for all $t \geq 0$

$$X_t = Y_t \;\text{ and }\; \rho_{t+1}(X) \leq \rho_{t+1}(Y) \;\;\Rightarrow\;\; \rho_t(X) \leq \rho_t(Y)$$

An equivalent characterization of TC is recursiveness:

$$\rho_t(X) = \rho_t(X_t 1_{\{t\}} - \rho_{t+1}(X) 1_{\{t+1,\ldots\}}) \quad \forall\; t \geq 0$$

Remark. $(\rho_t)_t$ on $\mathcal{R}^\infty$ is time consistent $\iff$ the corresponding $(\bar{\rho}_t)_t$ on $L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ is time consistent

Supermartingale properties

Let $(\rho_t)_t$ on $\mathcal{R}^\infty$ be continuous from above and **time consistent**. Then, $\forall\, \bar{Q} = Q \otimes D \ll \bar{P}$ such that $\alpha_0(Q \otimes D) < \infty$,

- the discounted penalty process $\quad (D_t \alpha_t(\bar{Q}))_{t \in \mathbb{T} \cap \mathbb{N}_0}$
- the 'global risk' process of $X \in \mathcal{R}^\infty$

$$D_t(\rho_t(X - X_t) + \alpha_t(\bar{Q})) - \sum_{s=0}^{t} D_s \Delta X_s, \quad t \in \mathbb{T} \cap \mathbb{N}_0$$

are Q-supermartingales.

Appearance of bubbles in the dynamic penalization

Riesz decomposition of the discounted penalty process:

$$D_t\alpha_t(\bar{Q}) = \underbrace{E_Q\Big[\sum_{k=t}^{T-1} D_k\alpha_{k,k+1}(\bar{Q})|\mathcal{F}_t\Big]}_{\text{“}\textit{fundamental penalization}\text{”}} + \underbrace{\lim_{s\to\infty} E_Q[D_s\alpha_s(\bar{Q})|\mathcal{F}_t]}_{\text{“}\textit{bubble}\text{”}} \quad Q\text{-a.s.}$$

↓

breakdown of asymptotic safety

where $\alpha_{k,k+1}$ is the ‘one-step’ penalty function, i.e., the penalty function of ρ_k restricted to the ‘one-step’ processes

$\rightarrow$ Bubbles reflect an excessive neglect of models which may be relevant for the risk assessment

Asymptotic safety

Consider $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, and fix a model $\bar{Q}$ s.t. $\alpha_0(\bar{Q}) < \infty$.

- $(\rho_t)_{t\in\mathbb{N}_0}$ on $\mathcal{R}^\infty$ is called **asymptotically safe** under the model $\bar{Q} = Q \otimes D$ if for any $X \in \mathcal{R}^\infty$

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X) \geq -X_\infty \quad Q\text{-a.s. on } \{D_\infty > 0\}$$

Theorem. For $(\rho_t)_{t\in\mathbb{N}_0}$ TC and continuous from above TFAE:

- $(\rho_t)_{t\in\mathbb{N}_0}$ is asymptotically safe under $\bar{Q}$;
- the model $\bar{Q}$ has no bubble, i.e., the martingale in the Riesz decomposition of $(D_t\alpha_t(\bar{Q}))_{t\in\mathbb{N}_0}$ vanishes.

Cash additivity and subadditivity

A conditional convex risk measure for processes ρ_t is called

- **cash subadditive** if for all $s > t$

$$\rho_t(X + m\mathbb{1}_{\{s,s+1,\dots\}}) \geq \rho_t(X) - m \qquad \forall m \in L^\infty_+(\mathcal{F}_t)$$
$$(\text{resp. } \leq \quad \forall m \in L^\infty_-(\mathcal{F}_t))$$

  (El Karoui & Ravanelli (2009))

- **cash additive at** s, for some $s > t$, if

$$\rho_t(X + m\mathbb{1}_{\{s,s+1,\dots\}}) = \rho_t(X) - m \qquad \forall m \in L^\infty(\mathcal{F}_t)$$

Remark. By monotonicity and cash-invariance every conditional convex risk measure for processes is cash subadditive

Time value of money

Proposition. Let $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ be continuous from above. Then

- ρ_t is **cash additive at time** $s > t \iff$ there is no discounting up to time s: $\forall\, \bar{Q} = Q \otimes D$ s.t. $\alpha_t(\bar{Q}) < \infty$

$$D_t = D_{t+1} = \cdots = D_s = 1 \quad Q\text{-a.s.}$$

- if $T < \infty$ or $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, ρ_t is **cash additive at all times** $s > t \iff$ it reduces to a risk measure for random variables:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left(E_Q[-X_T \mid \mathcal{F}_t] - \alpha_t(Q) \right)$$

- if $\mathbb{T} = \mathbb{N}_0$, ρ_t **cannot** be **cash additive at all times** $s > t$

Calibration to ZCB

- $(B_t)_{t=0,\dots,T}$, $B_t > 0\ \forall t$, money market account;
- zero coupon bonds for all maturities are available, with $B_{t,k}$ price at time t of a ZCB paying 1 at maturity k.

Suppose that ρ_t satisfies the following **calibration condition**:

$$\rho_t\left(\lambda_t \frac{B_t}{B_k} 1_{\{k,k+1,\dots\}}\right) = -\lambda_t B_{t,k} \quad \forall \lambda_t \in L^\infty(\mathcal{F}_t),\ k \geq t.$$

Then ρ_t is cash additive at time k if and only if

$$E_Q\left[\frac{B_t}{B_k} \,\middle|\, \mathcal{F}_t\right] = B_{t,k} \quad \forall Q\ :\ \exists D \text{ with } \alpha_t(Q \otimes D) < \infty$$

$\longrightarrow$ "no arbitrage" condition

Calibration to ZCB

In particular, if

- $(B_t)_{t=0,\dots,T}$ is predictable
- $(\rho_t)_{t=0,\dots,T}$ is time consistent

then ρ_t reduces to a convex risk measure on random variables $\forall t$.

That is, **discounting ambiguity is completely resolved** and we are only left with model ambiguity.

$\rightarrow$ the time value of the money is completely determined by the term structure specified by the prices of zero coupon bonds

Entropic risk measure for processes

On the product space the **conditional entropic risk measure** $\bar{\rho}_t : L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_T, \bar{P}) \to L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P})$ is defined by

$$\bar{\rho}_t(X) = \frac{1}{R_t} \cdot \log E_{\bar{P}} \left[e^{-R_t \cdot X} \;\middle|\; \bar{\mathcal{F}}_t \right]$$

with risk aversion parameter $R_t = (r_0, ..., r_{t-1}, r_t, ..., r_t)$, $r_s > 0$ and $\mathcal{F}_s$-measurable, for all $s = 0, ..., t$.

- The corresponding conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ takes the form

$$\rho_t(X) = \rho_t^{P,r_t}\Big(-\frac{1}{r_t} \log \Big(\sum_{s \geq t} e^{-r_t X_s} \mu_s^t \Big) \Big) = \rho_t^{P,r_t} \left(-\rho_t^{\mu(\omega), r_t(\omega)} \left(X_\cdot(\omega) \right) \right),$$

where ρ_t^{P,r_t} is the usual conditional entropic risk measure on random variables with risk aversion parameter r_t and $\rho^{\mu,r}$ is its analogous "with respect to time".

Average Value at Risk for processes

On the product space the **conditional Average Value at Risk** at level $\Lambda_t = (\lambda_0, ..., \lambda_{t-1}, \lambda_t, ..., \lambda_t)$, $0 < \lambda_s \leq 1$, $\lambda_s \in L^\infty(\mathcal{F}_s) \,\forall\, s$ is

$$\bar{\rho}_t(X) = \operatorname{ess\,sup}\{E_{\bar{Q}}[-X|\bar{\mathcal{F}}_t] \mid \bar{Q} \in \bar{\mathcal{Q}}_t, d\bar{Q}/d\bar{P} \leq \Lambda_t^{-1}\}$$

- The corresponding conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ takes the form

$$\rho_t(X) = \operatorname{ess\,sup}\left\{ E_Q\left[-\sum_{s \geq t} X_s \gamma_s \;\middle|\; \mathcal{F}_t \right] : \frac{\gamma_s M_s}{\mu_s^t} \leq \frac{1}{\lambda_t} \forall s \geq t \right\}$$

Unambiguous discounting process

If there is **no ambiguity** regarding the **discounting process**, i.e. $\exists\,!\,D \;\Rightarrow\;$ we can work on discounted terms:

$$Y_0 := X_0, \quad \Delta Y_s := D_s \Delta X_s \;\forall\, s \geq 1, \quad \text{and} \;\; Y_\infty := \lim_{t\to\infty} Y_t$$

Then ρ_t reduces to

$$\rho_t(X) = \psi_t\Big(\sum_{s=t}^{T} D_s \Delta X_s\Big) = \psi_t\Big(\sum_{s=t}^{T} \Delta Y_s\Big) = \psi_t(Y_T),$$

where $\psi_t : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ is a conditional convex risk measure for random variables.

Worst stopping

Let $\psi_t : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ be a conditional convex risk measure on random variables.

Θ_t = set of all stopping times valued in $\{t, t+1, ...\}$

Then $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\Omega, \mathcal{F}_t, P)$ defined by the worst stopping of $(\psi_t(X_s))_{s \geq t}$:

$$\rho_t(X) := \operatorname*{ess\,sup}_{\tau \in \Theta_t} \psi_t(X_\tau)$$

is a **convex risk measure on processes** (Cheridito & Kupper (2006)), with representation over the set of optional random measures

$$\left\{ (1_{\{\tau = s\}})_{s=t,t+1,...} | \tau \in \Theta_t \right\}$$

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