Optimal hedging in discrete and continuous time

Bruno Rémillard, HEC Montréal

Joint work with Sylvain Rubenthaler

Bachelier Finance Society

June 23rd 2010

Presentation plan

- Motivation
- Discrete time hedging
- Continuous time hedging
- Continuous time limit
- Example of application

Hedging problem

Goal: Find an optimal investing strategy for a portfolio

- Target: Payoff at maturity
- Investment strategy for the portfolio (optimal with respect to a measure of risk)
- Realtime implementation

Importance of hedging

Hedging is very important in finance as a tool for

- Option pricing
- Replication of hedge funds
- Risk management

Description of the problem

- S_k: Value of the d underlying assets at period k (assumed square integrable).
- $\mathbb{F} = \{\mathcal{F}_k, k = 0, \ldots, n\}$: Filtration. S is $\mathbb{F}$-adapted.
- $\Delta_k = \beta_k S_k - \beta_{k-1} S_{k-1}$, where the discounting factors β_k are predictable, i.e. β_k is $\mathcal{F}_{k-1}$-measurable for $k = 1, \ldots, n$.
- C: Payoff at period n.

Aim: Find an initial investment amount V_0 and a predictable investment strategy $\overrightarrow{\phi} = (\phi_k)_{k=1}^n$ that minimize the expected quadratic hedging error $E\left[\left\{G\left(V_0, \overrightarrow{\phi}\right)\right\}^2\right]$, where

$$G = G\left(V_0, \overrightarrow{\phi}\right) = \beta_n C - V_n,$$

and the discounted value of the portfolio at period k is

$$V_k = V_0 + \sum_{j=1}^{k} \phi_j^\top \Delta_j, \quad k = 0, \ldots, n.$$

Optimal hedging strategy

Set $P_{n+1} = 1$, and for $k = n, \ldots, 1$, define

$$
\begin{aligned}
A_k &= E\left(\Delta_k \Delta_k^\top P_{k+1} | \mathcal{F}_{k-1}\right), \\
b_k &= A_k^{-1} E\left(\Delta_k P_{k+1} | \mathcal{F}_{k-1}\right), \\
\alpha_k &= A_k^{-1} E\left(\beta_n C \Delta_k P_{k+1} | \mathcal{F}_{k-1}\right), \\
P_k &= \prod_{j=k}^{n} \left(1 - b_j^\top \Delta_j\right).
\end{aligned}
$$

Theorem

Suppose that $E(P_k|\mathcal{F}_{k-1}) \neq 0$ P-a.s., for $k = 1, \ldots, n$. Then the solution $\left(V_0, \overrightarrow{\phi}\right)$ of the minimization problem is $V_0 = E(\beta_n C P_1)/E(P_1)$, and

$$\phi_k = \alpha_k - V_{k-1} b_k, \quad k = 1, \ldots, n.$$

Option pricing

C_k: optimal investment at period k so that the value of the portfolio at period n is as close as possible to C, in terms of mean square error.

$$\Rightarrow \qquad \beta_k C_k = \frac{E(\beta_n C P_{k+1}|\mathcal{F}_k)}{E(P_{k+1}|\mathcal{F}_k)}, \qquad k = 0, \ldots, n.$$

Minimal martingale measure $\hat{P}$:

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_k} = \prod_{j=1}^{k} \frac{E(P_j|\mathcal{F}_j)}{E(P_j|\mathcal{F}_{j-1})}$$

Markovian dynamics

If the price process S is Markovian and $C_n = C_n(S_n)$, then $C_k = C_k(S_k)$, $\alpha_k = \alpha_k(S_{k-1})$, and $b_k = b_k(S_{k-1})$. It follows that all these functions can be approximated using the methodology developed in Papageorgiou et al. (2008).

Another interesting case encountered in practice is when S_k is not a Markov process but (S_k, h_k) is Markov, even if h_k is not observable, as in GARCH models or Hidden Markov models (HMM for short).

If $C_n = C_n(S_n)$, then $C_k = C_k(S_k, h_k)$,
$\alpha_k = \alpha_k(S_{k-1}, h_{k-1})$, and $b_k = b_k(S_{k-1}, h_{k-1})$.
Again, all these functions can be approximated using the methodology developed in Remillard et al. (2010).
Implementation of the hedging strategy then requires prediction of h_t given $S_0, \ldots, S_t$, which is a filtering problem.

Lévy processes

Examples

- Brownian motion
- Poisson process
- Jump-diffusion (Merton, 1976):

$$L_t = \mu t + \sigma B_t + \sum_{j=1}^{N_t} \zeta_j.$$

More generally a Lévy process L is a process with independent stationary increments, i.e.,

$$L_h, L_{2h} - L_h, \dots, L_{nh} - L_{(n-1)h}$$

are all independent and have the same distribution.

The only continuous Lévy processes are Brownian motions with drifts: $\mu t + \sigma B_t$.

In the following, we consider Lévy processes with exponential moments.

Representation

For the rest of the presentation, we only consider one dimensional processes. The multivariate case is treated in the paper.

A Lévy process L can be characterized by three parameters (μ, a, ν) such that for all $|\theta| \leq 2$,

$$E\left(e^{\theta L_t}\right) = e^{t\Psi_{\mu,a,\nu}(\theta)},$$

where

$$\Psi(\theta) = \theta\mu + \frac{1}{2}\theta^2 a + \int_{\mathbb{R}\setminus\{0\}} \left(e^{y\theta} - 1 - \theta y\right) \nu(dy).$$

Here $\mu \in \mathbb{R}$, $a > 0$ and ν is a Lévy measure. In particular, $E(L_t) = t\mu$, $\mathrm{Var}(L) = t(a + a_\nu)$, where $a_\nu = \int_{\mathbb{R}\setminus\{0\}} y^2\nu(dy)$.

Generator

Often financial models are described in terms of a stochastic differential equation.

Black-Scholes-Merton:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

A more practical approach is to describe the law of the process L through its infinitesimal generator $\mathcal{L}$: For all "nice" functions f,

$$f(x_t) - \int_0^t \mathcal{L} f(x_u) du$$

is a martingale. For a Lévy process with parameters (μ, a, ν),

$$\begin{aligned} \mathcal{L} f(x) &= \mu f'(x) + \frac{a}{2} f''(x) \\ &+ \int_{\mathbb{R} \setminus \{0\}} \{ f(x+y) - f(x) - y f'(x) \} \nu(dy). \end{aligned}$$

Examples

- Brownian motion: $\mathcal{L}f(x) = \frac{1}{2}f''(x)$.
- Poisson process with intensity λ:

$$\mathcal{L}f(x) = \lambda\{f(x+1) - f(x)\}, \quad x = 0, 1, \ldots$$

- Jump-diffusion:

$$\mathcal{L}f(x) = \mu f'(x) + \frac{\sigma^2}{2} f''(x) + \lambda \int \{f(x+y) - f(x)\} g(y) dy,$$

if the size of the jumps ζ_j have density g.

Regime-switching geometric Lévy processes

Given a regime-switching Lévy process L, process S, hereafter called a regime-switching geometric Lévy process,

$$S_t = se^{L_t}$$

is the associated regime-switching geometric Lévy process, i.e., (S, τ) is a Markov process with generator $\mathcal{L}$

$$\mathcal{L}f(s,i) = \mathcal{L}_i f(s,i) + \sum_{j=1}^{l} \Lambda_{ij} f(s,j),$$

where for each $i = 1, \ldots, l$, $\mathcal{L}_i$ is the generator of the geometric Lévy process $S_{i,t} = se^{L_{i,t}}$, and

$$\begin{aligned}\mathcal{L}_i f(s) &= s\psi(i)f'(s) + s^2 \frac{a(i)}{2} f''(s) \\ &\int_{\mathbb{R}\setminus\{0\}} \left[f\left\{s(1+y)\right\} - f(s) - ysf'(s)\right] \tilde{\nu}_i(dy),\end{aligned}$$

Set

$$(\Lambda_t)_{ij} = \Lambda_{ij}\gamma(t,j)/\gamma(t,i), \quad i \neq j,$$
$$(\Lambda_t)_{ii} = -\sum_{j\neq i}(\Lambda_t)_{ij},$$

where

$$\frac{d}{dt}\gamma(t,i) = -\ell(i)\gamma(t,i) + \sum_{j=1}^{I}\Lambda_{ij}\gamma(t,j), \quad \gamma(0,i) = 1,$$

$i = 1, \ldots, I.$
Λ_t is the generator of a time non homogeneous Markov chain $\tilde{\tau}$.

Extended Black-Scholes formula

Let C is the unique solution of

$$\partial_t C_t(s,i) + \mathcal{H}_{T-t} C_t(s,i) = r C_t(s,i), \quad C_T(s,i) = \Phi(s),$$

where

$$\mathcal{H}_t f(s,i) = rsf'(s,i) + \frac{a(i)}{2} s^2 f''(s,i) + \sum_{j=1}^{l} (\Lambda_t)_{ij} f(s,j)$$
$$+ \int \{1 - \rho(i) y\} \left[f\{s(1+y)\} - f(s) - ysf'(s) \right] \tilde{\nu}_i(dy).$$

Set

$$\alpha(t,s,i) \quad = \quad \partial_s C_t(s,i) + \frac{1}{s\mathbb{A}(i)} \{ C_t(s,i) m(i) + \mathcal{K}_i C_t(s,i) \},$$

where $\mathcal{K}_i f(s) = \int y \left[f\{s(1+y)\} - f(s) - ysf'(s) \right] \tilde{\nu}_i(dy)$.

Solution for regime-switching geometric Lévy processes

Explicit representation of the "Minimal Martingale Measure".

Theorem

The optimal solution of the hedging problem for a regime-switching geometric Lévy process is given by ϕ, and the actualized value of the associated portfolio is V,

where V satisfies the stochastic differential equation

$$V_t = C(0, s, i) + \int_0^t \alpha(u-, S_{u-}, \tau_{u-}) dX_u - \int_0^t V_{u-} dM_u$$

and $\phi_t = \alpha(t, S_{t-}, \tau_{t-}) - V_{t-} \frac{\rho(\tau_{t-})}{X_{t-}}$, with C and α defined below.

Martingale and change of measure

One can write

$$C_t(S_t, \tau_t) = E\left\{\Phi(S_T) Z_T | \mathcal{F}_t\right\} / \gamma_{T-t}(\tau_t),$$

where $M_t = \int_0^t \frac{\rho(\tau_{u-})}{X_{u-}} dX_u$ and $Z = \mathcal{E}\left\{-M\right\}$.

If Z is positive, then $\frac{d\hat{P}_i}{dP_i} = Z_T / \gamma(T, i)$ defines a change of measure under which X is a martingale.

For example, for the regime-switching geometric Brownian motion, S is continuous so Z is positive, being an exponential.

If Z is not positive, then the "price" $C_t(s, i)$ does not correspond to an expectation under an equivalent martingale measure.

Regime-switching Brownian motion

For that model $\nu_i \equiv 0$ and $\mathbb{A} = a$, S is continuous, and its generator is

$$\mathcal{L}f(s,i) = \psi(i) s f'(s,i) + \frac{a(i)}{2} s^2 f''(s,i) + \sum_{j=1}^{l} \Lambda_{ij} f(s,j).$$

It follows that

$$\mathcal{H}_t f(s,i) = r s f'(s,i) + \frac{a(i)}{2} s^2 f''(s,i) + \sum_{j=1}^{l} (\Lambda_t)_{ij} f(s,j)$$

is the generator of a time non homogeneous Markov process $(\tilde{S}, \tilde{\tau})$, where the Markov chain $\tilde{\tau}$ has generator (Λ_t), so

$$C_t(s,i) = e^{-r(T-t)} E\left\{\Phi(\tilde{S}_T) | \tilde{S}_t = s, \tilde{\tau}_t = i\right\}.$$

Next,

$$\alpha(t,s,i) = \partial_s C_t(s,i) + C_t(s,i)\rho(i)/s, \quad i = 1,\ldots,l.$$

Using the "pathwise method" in Broadie and Glasserman (1996), one can use simulations to obtain an unbiased estimate of α_t.

In fact if Φ is differentiable almost everywhere, then

$$\partial_s C_t(s,i) = \frac{1}{s} e^{-r(T-t)} E\left\{ \tilde{S}_T \Phi'(\tilde{S}_T) | \tilde{S}_t = s, \tilde{\tau}_t = i \right\},$$

so α_t can be written as an expectation of a function of $\tilde{S}_T$.

Finally,

$$\phi_t = \partial_s C_t(S_t, \tau_{t-}) + \left\{ C_t(S_t, \tau_{t-}) - e^{rt} V_{t-} \right\} \frac{\rho(\tau_{t-})}{S_t}.$$

In particular, $\phi_0 = \partial_s C_0(S_0, \tau_0)$. It follows that ϕ_t can be estimated by Monte-Carlo methods.

Optimal hedging vs delta-hedging

For the Black-Scholes-Merton model, there is perfect hedging, i.e., $V_t = e^{-rt} C_t(S_t)$, so $\phi_t = \partial_s C_t(S_t)$.

Its follows that the optimal hedging is delta-hedging only when there is no hedging error.

The formula

$$\phi_t = \partial_s C_t(S_t, \tau_{t-}) + \left\{ C_t(S_t, \tau_{t-}) - e^{rt} V_{t-} \right\} \frac{\rho(\tau_{t-})}{S_t}$$

allows for a "correction", using the hedging error $G_t = C_t(S_t, \tau_{t-}) - e^{rt} V_{t-}$.

Continuous time approximation

It can be shown that the discrete time regime-switching models can be approximated by their continuous time counterpart. Here we state some conditions under which the HMM model "converges " in some sense to a regime-switching geometric Lévy process.

More direct approach than in Prigent (2003).

Under slightly the same conditions, the "option prices" and the optimal strategy under a HMM model also converge in some sense to the optimal strategy of the regime-switching geometric Lévy process.

Continuous time limit of the HMM price process

Suppose now that for each n, one has a HMM model $\left(S_k^{(n)}, \tau_k^{(n)}\right)$, where $\beta_k^{(n)} = e^{-rTk/n}$. Define $S^{(n)}(t) = S^{(n)}_{[nt/T]}$.

From now on, when talking of convergence in law, denoted by $\rightsquigarrow$, we mean convergence in law in the space in the space of càdlàg functions over $[0, T]$ with the Skorohod topology.

For simplicity, let $\mathbb{E}_i$ denote expectation under the law of $\xi_1^{(n)}$ given $\tau_1^{(n)} = i$ and recall the following notations: $\mathbb{E}_i\left(\xi_1^{(n)}\right) = \mu^{(n)}(i)$ and $\mathbb{E}_i\left\{\left(\xi_1^{(n)}\right)^2\right\} = B^{(n)}(i)$, $i = 1\ldots, l$. Further let $C_2(\mathbb{R}^d)$ be the set of continuous functions f on $\mathbb{R}^d$ so that $f(y) = O(|y|^2)$ and $f(y)/|y|^2 \to 0$ as $y \to 0$.

Theorem

Suppose that $\lim_{n\to\infty} n\left(Q^{(n)} - I\right) \to \Lambda T$*. Assume also that for any* $i = 1, \ldots, l$*, the following conditions are satisfied, as* $n \to \infty$*:* $n\mu^{(n)}(i) \to Tm(i)$*,* $nB^{(n)}(i) \to T\mathbb{A}(i)$*, and for all* $f \in C_2(\mathbb{R}^d)$*,* $n\mathbb{E}_i\left\{f\left(\xi_1^{(n)}\right)\right\} \to T\int f(y)\tilde{\nu}_i(dy)$*.*

Then $\left(S^{(n)}, \tau^{(n)}\right) \rightsquigarrow (S, \tau)$ *with generator*

$$\mathcal{L}f(s,i) = \mathcal{L}_i f(s,i) + \sum_{j=1}^{l} \Lambda_{ij} f(s,j),$$

where for each $i = 1, \ldots, l$ *,*

$$\begin{aligned}\mathcal{L}_i f(s) &= s\psi(i)f'(s) + s^2\frac{a(i)}{2}f''(s) \\ &\quad \int_{\mathbb{R}\setminus\{0\}} \left[f\left\{s(1+y)\right\} - f(s) - ysf'(s)\right]\tilde{\nu}_i(dy),\end{aligned}$$

is the generator of a geometric Lévy process.

Example

Consider a regime-switching geometric Gaussian random walk with

$$\xi_k^{(n)} = e^{R_k^{(n)} - rT/n} - 1,$$

where under $\mathbb{P}_i$, $R_k^{(n)}$ *is Gaussian with mean* $\left\{\psi(i) - \frac{a(i)}{2}\right\} T/n$ *and variance* $a(i)T/n$.

It is easy to check that the conditions of the previous theorem are met with $\psi(i)$, $\mathbb{A}(i) = a(i)$ *and* $\nu_i \equiv 0$.

In other words, the limiting process is a regime-switching geometric Brownian.

Continuous time limit of the optimal hedging strategy

Suppose that the assumptions of the previous theorem are met.

Theorem

Suppose that $\Phi(s) = O(|s|^p)$, Φ *is almost everywhere differentiable with derivative* $\Phi'(s) = O(|s|^{p-1})$ *and* $E\left\{\left(\zeta^{(n)}\right)^k\right\} = 1 + \theta_k/n + o(1/n)$, $k = 1, \ldots, 2p+2$. *Then*

$$\left(S^{(n)}, \tau^{(n)}, C^{(n)}, \alpha^{(n)}, V^{(n)}, \phi^{(n)}\right) \rightsquigarrow (S, \tau, C, \alpha, V, \phi) \, .$$

For regime-switching geometric Gaussian random walk, the condition hold for call and put options with $p = 1$.

Call and put options on the S&P 500

Example comes from Remillard et al. (2010) where the authors analyzed the daily log-returns of the S&P 500 from January 1st 2007 to December 31st 2008.

They concluded that a regime-switching geometric Gaussian random walk with 4 regimes was the best fit for that data set.

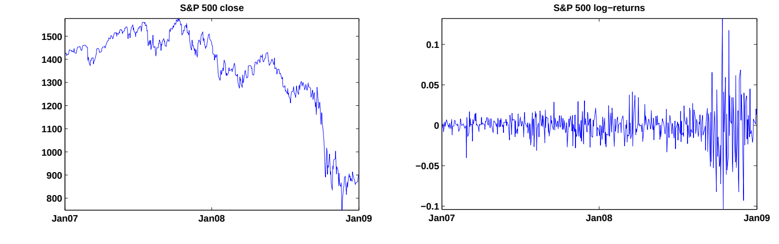

Figure: S&P 500 over the period 01/01/2007 to 12/31/2008.

Estimated parameters

Table: Parameter estimations of the daily log-returns using 4 regimes.

Regime	Mean	Variance	stat. distr.	Prob. of next regime
1	-0.00500	0.002221	0.133	0.0084
2	-0.00134	0.000191	0.517	0.9850
3	0.00131	0.000126	0.113	4.2798e-006
4	0.00119	0.000014	0.237	0.0064

Table: Transition matrix Q for 4 regimes.

Regime	1	2	3	4
1	0.9842	0.0158	0	0
2	0.0043	0.9744	0	0.0213
3	0	0	0	1
4	0	0.0542	0.4754	0.4704

From discrete case to continuous case

To find the associated parameters in continuous time (measured in years), one can multiply the mean and variance by 250 and set $\Lambda = 250(Q - I)$.

Our aim is to price, using a regime-switching geometric Brownian motion, at-the-money call and put options with a maturity of 0.12 years (30 days), using an annual rate of 3% and a starting price of the underlying asset of 100.

Parameters for the regime-switching geometric Brownian motion

Table: Parameters for the continuous time case.

Regime	ψ	A	ρ	ℓ
1	-0.9724	0.5553	-1.8053	1.8096
2	-0.3111	0.0478	-7.1440	2.4370
3	0.3433	0.0315	9.9444	3.1151
4	0.2993	0.0035	76.9286	20.7130

Table: Generator Λ.

Regime	1	2	3	4
1	-3.9500	3.9500	0	0
2	1.0750	-6.4000	0	5.3250
3	0	0	-250.0000	250.0000
4	0	13.5500	118.8500	-132.4000

Simulation results

The next table contains prices of at-the-money call and put options, together with the value of $\phi_0 = \partial_s C_0(s, i)$, obtained by using 1,000,000 repetitions and antithetic variables.

Using previous results, one predicts that the next regime will be regime 2, having probability .98.

Because one can evaluate C_t and ϕ_t for any t, one could do as proposed in Remillard et al. (2010) and compare the optimal discrete hedging with the discretized version, i.e., by considering $\phi_{Tk/n}$ for $k = 1, \ldots, n$, as in the discretized version of the Black-Scholes model, using filtering to predict the regimes using information available previously.

95% confidence intervals for the price of at-the-money calls and puts, together with initial investments, using 1,000,000 simulations.

	Call	
Regime	Price	ϕ_0
1	9.3103 ± 0.0182	0.5524 ± 0.0004
2	3.5034 ± 0.0069	0.5356 ± 0.0001
3	2.6398 ± 0.0049	0.5380 ± 0.0002
4	2.6469 ± 0.0049	0.5384 ± 0.0002

	Put	
Regime	Price	ϕ_0
1	8.9549 ± 0.0110	-0.4475 ± 0.0003
2	3.1435 ± 0.0055	-0.4644 ± 0.0001
3	2.2803 ± 0.0041	-0.4620 ± 0.0002
4	2.2874 ± 0.0042	-0.4616 ± 0.0002

References I

Broadie, M. and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Management Science*, 42:260–285.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144.

Papageorgiou, N., Rémillard, B., and Hocquard, A. (2008). Replicating the properties of hedge fund returns. *Journal of Alternative Invesments*, 11:8–38.

Prigent, J.-L. (2003). *Weak convergence of financial markets*. Springer Finance. Springer-Verlag, Berlin.

Remillard, B., Hocquard, A., and Papageorgiou, N. A. (2010). Option Pricing and Dynamic Discrete Time Hedging for Regime-Switching Geometric Random Walks Models. Working paper series.