Exotic Options in Multiple Priors Models

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Outline

- Motivation
- Mathematical Framework
- Exotic Options in Multiple Priors Models
  - Dual Expiry Options
  - Piecewise monotone Payoffs
- Conclusions and future Work

Motivation

Classical Exercise Problem for American Options

American options:

- The right to buy or sell an underlying S at any time prior to maturity T subject to a contract
- Realizing the profit $A(t, (S_s)_{s \leq t})$ when exercised at t

Problem of the buyer:

- Exercise the option optimally choosing a strategy that maximizes the expected reward of the option, i.e. choose a stopping time τ^* that maximizes

$$\mathbb{E}^P((A(\tau, (S_s)_{s \leq \tau}))) \text{ over all stopping times } \tau \leq T$$

under an appropriately chosen measure P

How to choose P?

- P is the equivalent martingale measure in complete markets
- P is the physical measure in real option models

Solution

- For fixed stochastic basis backward induction leads to the solution
- Snell envelope defines the value function of the problem through

$$U_T = A(T,(S_s)_{s\leq T})/(1+r)^T$$
$$U_t = \max\{A(t,(S_s)_{s\leq t})/(1+r)^t, \mathbb{E}^P(U_{t+1}|\mathcal{F}_t)\}$$

for $t < T$

- Stop as soon as the value process reaches the payoff process

Motivation for Multiple Priors Models

- What is if the market is imperfect?
- Information is imprecise?
- Regulation imposes constraints on trading rules?

Several answers are possible:

- Superhedging
- Utility indifference pricing
- Risk measure pricing

Our approach:

- Ambiguity pricing

Aim of the paper

Ambiguity pricing

- Take the perspective of a decision maker who is uncertain about the underlying's dynamics and uses a set of priors instead of a single one
- Being pessimistic she maximizes the lowest expected return of option

$$\text{maximize} \quad \inf_{P \in \mathcal{P}} \mathbb{E}^P (A(\tau, S_\tau)/(1+r)^\tau)$$

- Concentrate on the effect of ambiguity and assume risk neutrality
- Model a consistent market under multiple priors assumption
- Study several exotic options of American style in the framework of ambiguity pricing
- Analyze the difference between classical expected return based pricing and the coherent risk pricing

Results

Economically

- Ambiguity pricing leads to a valuation under a specific pricing measure
- The pricing measure is rather a part of the solution then of the model itself
- The pricing measure captures the fears of the decision maker and depends on the state and the payoff structure

Mathematically

- The pricing measure might looses the independence property
- Cut off rules are still optimal in this model
- The use of the worst-case measure increases the complexity

General Framework

The Mathematical Setup

- **A probability space** $(\Omega, \mathcal{F}, \mathbb{P}_0)$
  - $\Omega = \otimes_{t=1}^{T}\{0,1\}$ – the set of sequences with values in $\{0,1\}$
  - $\mathcal{F}$ – the σ-field generated by all projections $\epsilon_t : \Omega \to \{0,1\}$
  - $\mathbb{P}_0$ – the uniform on $(\Omega, \mathcal{F})$
- **A filtration** $(\mathcal{F}_t)_{t=0,\ldots,T}$ generated by the sequence $\epsilon_1, \ldots, \epsilon_t$ with $\mathcal{F}_t = \sigma(\epsilon_1, \ldots, \epsilon_t)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{F}_T$

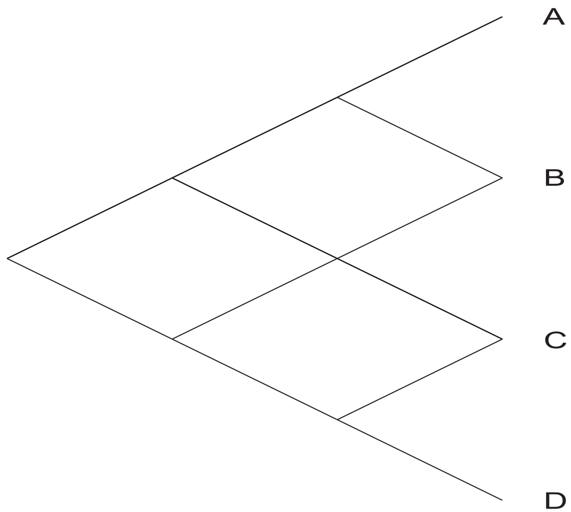


Figure 1: Binomial tree

The Mathematical Setup

- **A convex set of priors** $\mathcal{P}$ defined via

$$\mathcal{P} = \left\{ P \in \Delta(\Omega, \mathcal{F}) | P(\epsilon_t = 1 | \mathcal{F}_{t-1}) \in [\underline{p}, \overline{p}] \ \forall t \leq T \right\}$$

for a fixed interval $[\underline{p}, \overline{p}] \subset (0, 1)$

- $\mathcal{P}$ contains all product measures defined via $P_p(\epsilon_{t+1} = 1 | \mathcal{F}_t) = p$ for a fixed $p \in [\underline{p}, \overline{p}]$ and all $t \leq T$
- Denote by $\overline{P}$ the measure $P_{\overline{p}}$ and by $\underline{P}$ the measure $P_{\underline{p}}$
- $\epsilon_1, \ldots, \epsilon_t$ are i.i.d under all product measures $P_p \in \mathcal{P}$
- In general, no independence

Lemma 1 *The above defined set of priors $\mathcal{P}$ satisfies*

1. *For all $P \in \mathcal{P} \quad P \sim \mathbb{P}_0$*

- All measures in $\mathcal{P}$ agree on the null sets
- We can identify $\mathcal{P}$ with the set of density processes $\mathcal{D} = \{\mathcal{D}_t | t \leq T\}$ where

$$\mathcal{D}_t = \left\{ \frac{dP}{d\mathbb{P}_0}\bigg|_{\mathcal{F}_t} | P \in \mathcal{P} \right\}$$

- $\inf$ is always a $\min$

Lemma 2 *$\mathcal{P}$ is time-consistent in the following sense: Let $P, Q \in \mathcal{P}$, $(p_t)_t, (q_t)_t \in (\mathcal{D}_t)_t$. For a fixed stopping time $\tau \leq T$ define the measure R via*

$$r_t = \begin{cases} p_t & \text{if } t \leq \tau \\ \frac{p_\tau q_t}{q_\tau} & \text{else} \end{cases}$$

Then $R \in \mathcal{P}$.

Time-consistency is equivalent to

- a version of *The Law of Iterated Expectations*
- fork-stability (Föllmer/Schied (2004))
- rectangularity (Epstein/Schneider (2003))

$\Rightarrow$ Allows to change the measure between periods

Ambiguous version of the Cox–Ross–Rubinstein model

- A market with 2 assets:
  - A riskless asset B with interest rate $r > 0$
  - A risky asset S evolving according to $S_0 = 1$ and

$$S_{t+1} = \begin{cases} S_t \cdot u & \text{if } \epsilon_{t+1} = 1 \\ S_t \cdot d & \text{if } \epsilon_{t+1} = 0 \end{cases}$$

- Assume $u \cdot d = 1$ and $0 < d < 1 + r < u$
- $\overline{P}/\underline{P}$ is the measure with the highest/lowest mean return
- Path-dependent increments
- Dynamical model adjustment without learning

The Decision Problem

Exercise problem of an ambiguity averse buyer

- For an option paying off $A(t,(S_s)_{s\le t})$ when exercised at t:
- Choose a stopping time τ^* that maximizes

$$\min_{P\in\mathcal{P}} \mathbb{E}^P\left(A(\tau,(S_s)_{s\le\tau})/(1+r)^\tau\right)$$

over all stopping times $\tau \le T$

- Compute

$$U_t^{\mathcal{P}} = \operatorname*{esssup}_{\tau\ge t} \operatorname*{essinf}_{P\in\mathcal{P}} \mathbb{E}^P\left(A(\tau,(S_s)_{s\le\tau})/(1+r)^\tau|\mathcal{F}_t\right)$$

– the ambiguity value of the claim at time t

Theorem 1 (Riedel (2009)) *Given a set of measures $\mathcal{P}$ as above and a bounded payoff process X, $X_t = A(t,(S_s)_{s\le t})/(1+r)^t$, define the* **multiple priors Snell envelope** $U^{\mathcal{P}}$ *recursively by*

$$U_T^{\mathcal{P}} = X_T \tag{1}$$
$$U_t^{\mathcal{P}} = \max\{X_t, \operatorname*{essinf}_{P\in\mathcal{P}} \mathbb{E}^P(U_{t+1}^{\mathcal{P}}|\mathcal{F}_t)\} \text{ for } t < T$$

Then,

1. $U^{\mathcal{P}}$ *is the value process of the multiple priors stopping problem for the payoff process* X*, i.e.*

$$U_t^{\mathcal{P}} = \operatorname*{esssup}_{\tau\ge t} \operatorname*{essinf}_{P\in\mathcal{P}} \mathbb{E}^P(X_\tau|\mathcal{F}_t)$$

2. *An optimal stopping rule is then given by*

$$\tau^* = \inf\{t\ge 0 | U_t^{\mathcal{P}} = X_t\}$$

Duality result (KARATZAS/ KOU (1998)): There exists a $\hat{P} \in \mathcal{P}$ s.t.

$$U^{\mathcal{P}} = U^{\hat{P}} \quad \mathbb{P}_0 - \text{a.s.}$$

To solve the problem

- Identify the worst-case measure $\hat{P} \in \mathcal{P}$
- Refer to the classical solution

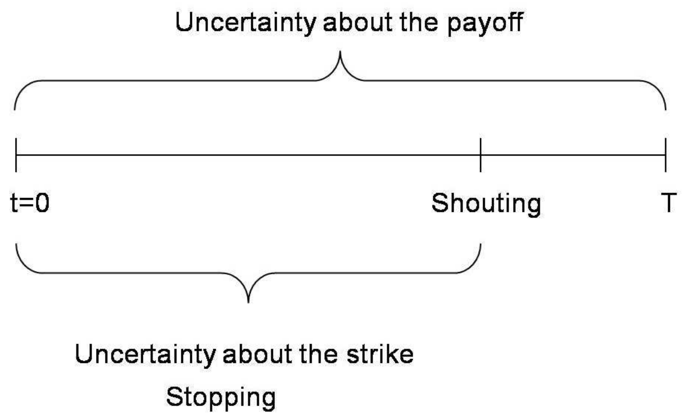
Idea

- Identify the worst-case measure for monotone claims
- Decompose more complicated claims in monotone parts
- Construct the worst-case measure pasting together the worst-case densities of the monotone parts

Exotic Options in Multiple priors Models

Multiple Expiry Options

- **Multiple expiry options** expiry at some date $\sigma < T$ in the future issuing a new option with conditions specified at $\sigma < T$
- Often used as employee bonus and therefore are subject to trading restrictions
- The value to the buyer/executive differs from the cost to the company of granting the option (HALL/ MURPHY (2002))
- Multiple expiry feature causes a second source of uncertainty:



Dual Expiry Options – Shout Options

- **Shout options** allow the buyer to shout and freeze the strike at-the-money at any time prior to maturity
- Can be seen as the option to abandon a project to conditions specified by the buyer
- There is uncertainty about the strike at time 0 that is resolved at the time of shouting
- The payoff of the shout option at shouting is an at-the-money put of European style and the problem becomes

$$\text{maximize } A(\sigma, S_\sigma) = (S_\sigma - S_T)^+/(1+r)^T$$

over all stopping times $\sigma \leq T$

- The task here is rather to start the process optimally than to stop it

- Since the payoff process is not adapted consider for $t \leq T$

$$
\begin{aligned}
X_t =& \operatorname*{essinf}_{P \in \mathcal{P}} \mathbb{E}^P\left((S_t - S_T)^+/(1+r)^T | \mathcal{F}_t\right) \\
=& S_t \cdot g(t, \overline{P}) \\
=& S_t \cdot \frac{(1-\overline{p})^T}{(1+r)^T} \left(\sum_{k=0}^{k(t)} \binom{T-t}{k} \left(\frac{\overline{p}}{1-\overline{p}}\right)^k (1 - d^{T-2k}) \right)
\end{aligned}
$$

for $k(t) = \left\lfloor \frac{T-t}{2} \right\rfloor$

Lemma 3 *For all stopping times $\sigma \leq T$ we have*

$$
\min_{P \in \mathcal{P}} \mathbb{E}^P(X_\sigma) = \min_{P \in \mathcal{P}} \mathbb{E}^P(A(\sigma, S_\sigma)/(1+r)^T)
$$

Dual Expiry Options – Shout Floor

- We can maximize X instead of the original payoff
- As a consequence we have

$$
\begin{aligned}
U_0^{\mathcal{P}} &= \operatorname*{essinf}_{P\in\mathcal{P}} \mathbb{E}^P \left(\operatorname*{essinf}_{Q\in\mathcal{P}} \mathbb{E}^Q((S_{\sigma^*} - S_T)^+ | \mathcal{F}_{\sigma^*}) \right) \\
&= \min_{P\in\mathcal{P}} \mathbb{E}^P \left(S_{\sigma^*} \cdot g(\sigma^*, \overline{P}) \right) \\
&= \mathbb{E}^{\underline{P}} \left(S_{\sigma^*} \cdot g(\sigma^*, \overline{P}) \right)
\end{aligned}
$$

where σ^* is optimal.

- The worst-case measure is defined by

$$
\hat{P}(\epsilon_{t+1} | \mathcal{F}_t) = \begin{cases} \overline{p} & \text{if } \sigma^* < t \\ \underline{p} & \text{else} \end{cases}
$$

Dual Expiry Options – Shout Floor

Lemma 4 *The optimal stopping time for the above problem is given by*

$$\sigma^* = \inf\{t \geq 0 : f(t) = x^*\}$$

where

$$f(t) = g(t, \overline{P}) \cdot (\underline{p} \cdot u + (1 - \underline{p}) \cdot d)^t$$

and x^* *is the maximum of* f *on* $[0, T]$

Proof: Generalized parking method and Optional Sampling

Remarks

1. Closed form solutions require exact study of the monotonicity of f
2. $1 - \overline{p} \geq (\underline{p} \cdot u + (1 - \underline{p}) \cdot d)$ is sufficient to have

$$\sigma^* = 0$$

U–shaped Payoffs

- **U–shaped payoffs** consist of two monotone parts allowing to benefit from change in the underlying independently of the direction of the change
- Often used as speculative instrument before important events

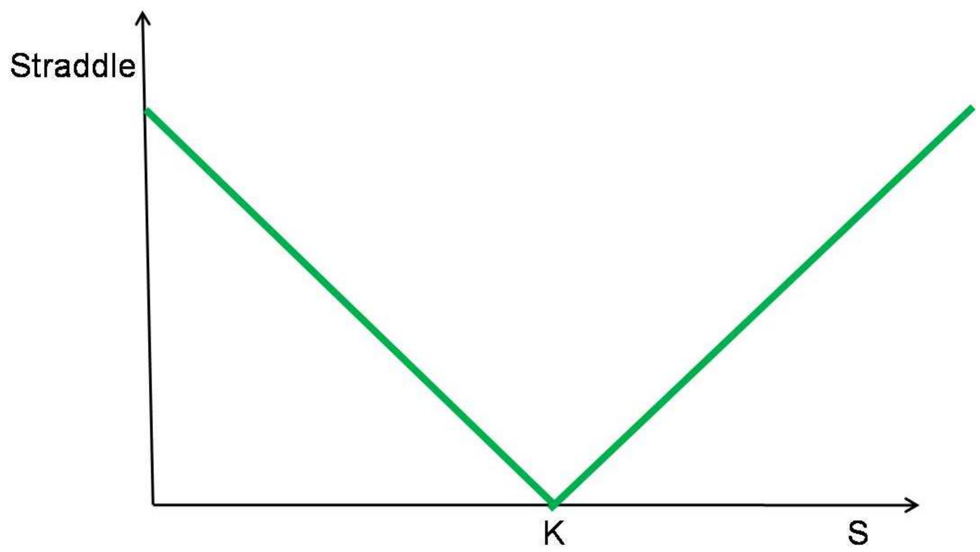


Figure 2: Payoff of Straddle

U–shaped Payoffs

- Up- and Down-movement can increase the value of the claim
- Uncertainty does not vanish over time

Lemma 5 *The value process is Markovian. For every $t \leq T$ the value function $v(t,\cdot)$ is quasi-convex and there exists a sequence $(\hat{x}_t)_{t\leq T}$ s.t. $v(t,\cdot)$ increases on $\{x_t > \hat{x}_t\}$ and decreases else.*

Proof: Backward induction

- Proof uses explicitly the binomial structure of the model
- As a consequence we obtain

$$\hat{P}(\epsilon_{t+1} = 1|\mathcal{F}_t) = \begin{cases} \overline{p} & \text{if } S_t < \hat{x}_t \\ \underline{p} & \text{if } S_t \geq \hat{x}_t \end{cases}$$

U–shaped Payoffs

- The worst-case measure is mean-reverting (in a wider sense)
- The drift changes every time S hits a barrier and can happen arbitrary often
- Fears of the decision maker are opposite to the market movement
- Increments are not independent anymore
- Idea: Use the generalized parking method again or upper and lower bounds

Conclusions

Conclusions

- A model of a multiple priors market provided
- A method to evaluate options in imperfect markets proposed
- Pricing measure for several classes of payoffs derived
- Worst-case measure is path-dependent in general
- The structure of the stopping times carries over in this model
- This is, however, due to the model and not a general result

Future work

- Continuous-time analysis – Brownian motion setting
- Infinite time modeling in continuous time
  - Allows for closed form solutions and comparative statics
  - Mathematical traps due to multiple measure structure
  - More modeling necessary to build a meaningful mathematical model

Thank you for your attention!