Pricing Options on Variance in Affine Stochastic Volatility Models

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6th World Congress of the Bachelier Finance Society

Toronto, June 23, 2010
Outline

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Summary
Realized variance of a stock $S = S_0 \exp(X)$ for fixings $0 = t_0 < \ldots < t_N = T$:

$$\sum_{n=1}^{N} \log\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)^2 = \sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2$$

Options on variance:
- Variance swap
- Volatility swap
- Puts on variance, variance calls, etc.

Tractable pricing formulas in realistic models?
Introduction

Quadratic variation

For $\sup_{n=1,\ldots,N} |t_n - t_{n-1}| \to 0$:

$$\sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2 \to [X, X]_T$$

in probability

- Sepp (2008), Broadie & Jain (2008): Typically good approximation via **quadratic variation** $[X, X]$ for daily fixings
- Exception: Short-dated call options
- Here: Use approximation via quadratic variation $[X, X]$
- What type of structure of $X$ makes this tractable?
Introduction

Literature

For **continuous** stock prices **without leverage**:
- Benth et al. (2007): BNS model

Models with **jumps**:
- Carr et al. (2005): Lévy processes
- Sepp (2008), Broadie & Jain (2008): Heston models with specific compound Poisson jumps
- Carr & Itkin (2009): Options on predictable quadratic variation \( \langle X, X \rangle \) in time-changed Lévy models

**Unifying framework** including jumps, stochastic volatility and the leverage effect?
Introduction

Fourier-Laplace methods

Carr & Madan (1999), Raible (2000): Consider European-style option (e.g. put, call) with payoff

\[ f(X_T) = \int_{R-i\infty}^{R+i\infty} l(z)e^{zX_T} \, dz, \quad R \in \mathbb{R} \]

\[ E_Q[f(X_T)] = \int_{R-i\infty}^{R+i\infty} l(z)E_Q[e^{zX_T}] \, dz \]

- Price under risk-neutral measure \( Q \) given by

- Tractable via numerical quadrature, if Fourier-Laplace transform \( E_Q[e^{zX_T}] \) is known, likewise for \( [X, X] \)

- Flexible model class where this is the case: \textbf{Affine processes} characterized by Duffie et al. (2003)
Affine Stochastic Volatility Models

Definition

- Affine local characteristics of $X$ and volatility $\nu$:
  \[ b^{(\nu, X)} = \beta_0 + \beta_1 \nu, \quad c^{(\nu, X)} = \gamma_0 + \gamma_1 \nu, \]
  \[ K^{(\nu, X)}(dx) = \kappa_0(dx) + \kappa_1(dx)\nu \]

- Affine conditional Fourier-Laplace transform:
  \[ E[e^{zX_T}|F_t] = \exp(\psi_0(T - t, z) + \psi_1(T - t, z)\nu_t + zX_t), \]
  where \( \psi_0(t, z) = \int_t^T \psi_0(\psi_1(t, z), z)dt \) and
  \[ \partial_t \psi_1(t, z) = \psi_1(\psi_1(t, z), z), \quad \psi_1(0, z) = 0 \]

- Generalized Riccati PIDE with
  \[ \psi_i(z) = \beta_i^T z + \frac{1}{2} z^T \gamma_i z + \int (e^{zx} - 1 - zx) \kappa_i(dx) \]
Affine Stochastic Volatility Models

Examples

Includes most models from the option pricing literature:

- **Lévy models**
- **CIR-time-change models** (generalized Heston models):
  \[ X_t = L \int_0^t v_s ds + \rho (v_t - v_0) + \text{Drift} \]
  \[ dv_t = (\eta - \lambda v_t) dt + \sigma \sqrt{v_t} dZ_t \]
  for Lévy process \( L \), Wiener process \( Z \)

- **OU-time-change models** (generalized BNS models):
  \[ X_t = L \int_0^t v_s ds + \rho Z_t + \text{Drift} \]
  \[ dv_t = -\lambda v_t dt + dZ_t \]
  for Lévy process \( L \), subordinator \( Z \)
Affine Stochastic Volatility Models

Quadratic variation: Characterization

Definition:

\[
[X, X]_t = \langle X^c, X^c \rangle_t + \sum_{s \leq t} \Delta X_s^2
\]

Local characteristics:

\[
b^{[X,X]} = c^X + \int x^2 K^X(dx), \quad c^{[X,X]} = 0,
\]

\[
K^{[X,X]}(G) = \int 1_G(x^2) K^X(dx) \quad \forall G \in \mathcal{B}^2
\]

Key observation: \((\nu, X, [X, X])\) is affine in \(\nu\)!

Still analytically tractable, characteristic function via generalized Riccati equations

Compare \((r, \int_0^t r_t dt)\) in affine short-rate models
Affine Stochastic Volatility Models

Quadratic variation: Characteristic function

Fourier-Laplace transform of $[X, X]_T$:

- Need to solve generalized Riccati PIDE
- No quadratic term, since $[X, X]$ is of finite variation
- But need to evaluate terms of the form

$$\int (e^{zx^2} - 1 - zx^2)K^X(dx),$$

since $\Delta[X, X]_t = \Delta X^2_t$

- In many models of interest, this can be done using special functions
- Only difference compared to evaluation of stock options
- Then: Swaps via differentiation, options via integration
Example 1: Generalized Heston model of Carr et al. (2003):

\[ X_t = L \int_0^t v_s ds + \rho (v_t - v_0) + \text{Drift}, \quad dv_t = (\eta - \lambda v_t) dt + \sigma \sqrt{v_t} dZ_t \]

for Lévy process \( L \) with triplet \((b_L, c_L, K_L)\), Wiener process \( Z \).

Then:

\[
E\left[ e^{z[X,X]_T} \mid \mathcal{F}_t \right] = e^{\Psi_0(T-t,z)+\Psi_1(T-t,z)v_t+z[X,X]_t}
\]

- \( \Psi_1(t, z) = \frac{2g(z)(e^{\psi(z)}t - 1)}{f(z) - \lambda + e^{\psi(z)}t(f(z) + \lambda)} \)
- \( \Psi_0(t, z) = \frac{2\eta}{\sigma^2} \log \left( \frac{2f(z)e^{t(f(z) + \lambda)/2}}{f(z) - \lambda + e^{\psi(z)}t(f(z) + \lambda)} \right) \)
- \( f(z) = \sqrt{\lambda^2 - 2\sigma^2 g(z)} \), \( g(z) = (\sigma^2 \rho^2 + c_L) z + \int (e^{xz^2} - 1) K_L(dx) \)

typically known in terms of special functions

\[ dX_t = (\text{Drift}) dt + \sqrt{v_t} dW_t + \rho dZ_t, \quad dv_t = -\lambda v_t dt + dZ_t \]

for compound poisson process \( Z \) with rate \( a \) and \( \exp(b) \)-jumps. Then:

\[
E[e^{z[X,X]_T} | \mathcal{F}_t] = e^{\Psi_0(T-t,z) + \Psi_1(T-t,z)v_t + z[X,X]_t}
\]

- \( \Psi_1(t, z) = \frac{1 - e^{-\lambda t}}{\lambda} z \)

- \( \Psi_0(t, z) = \frac{ab}{2\sqrt{-\rho^2 z}} \int_0^t U \left( \frac{1}{2}, \frac{1}{2}, \frac{(b - \Psi_1(s,z))^2}{-4\rho^2 z} \right) ds - at \) for hypergeometric \( U \)-function

- One extra \( dt \)-integral compared to generalized Heston
Pricing Options on Variance

Variance swaps

- Choose swap rate $K_{\text{var}}$ such that
  \[ E_Q([X, X]_T - K_{\text{var}}) = 0 \]

- Differentiation of the characteristic function:
  \[ E_Q([X, X]_T | \mathcal{F}_t) = [X, X]_t + \partial_u \psi_0(T - t, 0) + \partial_u \psi_1(T - t, 0) \nu_t \]

- Variance swap dynamics are (inhomogeneously) affine!
- Opens the door to mean-variance hedging etc.
- Moreover: Explicit formulas for $K_{\text{var}}$ in concrete models, e.g.,
  \[ K_{\text{var}} = \left( \frac{e^{-\lambda T} - 1 + \lambda T}{\lambda^2} \right) \frac{a}{b} + \frac{2a \rho^2}{b^2} T + \frac{1 - e^{-\lambda T}}{\lambda} \nu_0 \]
  for BNS model from above
Pricing Options on Variance
European payoffs $f([X, X]_T)$

- Volatility swap: $f(x) = \sqrt{x} - K_{vol}$, hence
  $$K_{vol} = \frac{1}{2\sqrt{\pi}} \int_0^\infty 1 - E_Q[e^{-z[X, X]_T}] \frac{dz}{z^{3/2}}$$

- Put on variance: $f(x) = (K - x)^+$, hence
  $$E_Q[(K - x)^+] = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \frac{e^{-Kz}}{z^2} E_Q[e^{z[X, X]_T}] dz, \quad R < 0$$

- Evaluation via numerical quadrature
- Similar but simpler formulas for $\langle X, X \rangle$. No special functions, just one $dz$-integration
  $\Rightarrow$ Good approximation?
Numerical Illustration

Variance and volatility swaps

Above BNS model with calibrated parameters of Schoutens (2003):

Considerable difference between quadratic variation \([X, X]\) and its predictable counterpart \(\langle X, X \rangle\)
Numerical Illustration
Puts on variance

Above BNS model with calibrated parameters of Schoutens (2003):

> Again systematic error for approximation of $[X, X]$ with $\langle X, X \rangle$
Summary
Pricing options on variance in affine stochastic volatility models

▶ Approximation of realized variance by \([X, X]\)
▶ Affine structure of \((\nu, X)\) passed on to \((\nu, X, [X, X])\)
▶ Characteristic function via generalized Riccati equations
▶ Variance swap prices via differentiation, volatility swaps, puts, calls etc. via numerical quadrature
▶ Integrands somewhat more involved than for stock options (special functions!), but still tractable
▶ Price processes of variance swaps are inhomogeneously affine

For more details: