American-style options, stochastic volatility, and degenerate parabolic variational inequalities

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Introduction

- Degenerate Markov processes and their associated parabolic PDEs are pervasive in finance.
- Degenerate parabolic PDEs give rise to challenging terminal/boundary value problems (European-style options) and terminal/boundary value obstacle problems (American-style options).
- What boundary conditions are appropriate or necessary?

Degenerate elliptic and degenerate parabolic partial differential equations

- Research goes back to Kohn and Nirenberg (1965).
- A highly selective list includes Daskalopoulos and her collaborators, Feller, Freidlin, Koch, Kufner, Levendorskii, Opic, Pinsky, Stredulinsky, ...
- Although previous research on degenerate elliptic/parabolic PDEs is extensive, more often than not, results often exclude even simple examples of interest in finance (CIR, Heston, etc).
- Recent research due to Ekstrom and Tysk for CIR PDEs and Laurence and Salsa for solutions of American-style, multi-asset BSM option pricing problems.

Heston's Stochastic Volatility Process

Heston's asset price process, $S(u) = \exp(X(u))$, is defined by

$$dX(u) = (r - q - Y(u)/2))\, du + \sqrt{Y(u)}\, dW_1(u), \quad X(t) = x,$$
$$dY(u) = \kappa(\theta - Y(u))\, du + \sigma\sqrt{Y(u)}\, dW_2(u), \quad Y(t) = y,$$

where $(W_1(u), dW_3(u))$ is two-dimensional Brownian motion, $W_2(u) := \rho W_1(u) + \sqrt{1-\rho^2}\, W_3(u)$, κ, θ, σ are positive constants, $\rho \in (-1, 1)$, $r \geq 0$, $q \geq 0$, and $Y(u)$ is the variance process.

Degenerate parabolic PDEs and variational inequalities

Option pricing problems for the Heston process lead to

- *Degenerate* parabolic differential equations,
- *Degenerate* parabolic variational inequalities,

for European and American-style option pricing problems, respectively.

Heston parabolic differential equation

If $-\infty \leq x_0 < x_1 < \infty$, let $\mathscr{O} := (x_0, x_1) \times (0, \infty)$ and $Q := [0, T) \times \mathscr{O}$. If $\psi : Q \to \mathbb{R}$ is a suitable function, for example, $\psi(t, x, y) = (K - e^x)^+$ or $(e^x - K)^+$, and $r \geq 0$, define

$$u(t, x, y) := e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}^{t,x,y} \left[\psi(T, X(T), Y(T)) \right],$$

then we expect

$$-u' + Au = 0 \quad \text{on } Q, \quad u(T, \cdot) = \psi(T, \cdot) \quad \text{on } \mathscr{O},$$

where

$$-Au := \frac{y}{2} \left(u_{xx} + 2\rho\sigma u_{xy} + \sigma^2 u_{yy} \right) + (r - q - y/2) u_x + \kappa(\theta - y) u_y - ru.$$

Degenerate elliptic or parabolic PDEs

Suppose $(t, x) \in Q = [0, T) \times \mathscr{O}$ and $\mathscr{O} \subset \mathbb{R}^n$, and

$$
\begin{aligned}
-Au(t,x) := \frac{1}{2} \sum_{i,j} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) \\
+ \sum_i b_i(t,x) \frac{\partial u}{\partial x_i}(t,x) - c(t,x) u(t,x).
\end{aligned}
$$

If $\xi^T A(t,x)\xi \geq \mu(t,x)|\xi|^2$, $\xi \in \mathbb{R}^n$, where $\mu(x) > 0$, then A is *elliptic (parabolic) on* Q if $\mu > 0$ on Q, and A is *uniformly elliptic (parabolic) on* Q if $\mu \geq \delta$ on Q, for some constant $\delta > 0$. This condition fails for the Heston operator, as $\mu = 0$ along $\{y = 0\}$ component of $\bar{\mathscr{O}}$ and the operator is "degenerate".

Weighted Sobolev spaces

Definition

We need a weight function when defining our Sobolev spaces,

$$\mathfrak{w}(x,y) := \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x|-\mu y}, \quad \beta = \frac{2\kappa\theta}{\sigma^2}, \mu = \frac{2\kappa}{\sigma^2},$$

for $(x,y) \in \mathscr{O}$ and a suitable positive constant, γ. Then

$$H^1(\mathscr{O},\mathfrak{w}) := \{u \in L^2(\mathscr{O},\mathfrak{w}) : (1+y)^{1/2} u \in L^2(\mathscr{O},\mathfrak{w}), \\ \text{and } y^{1/2} Du \in L^2(\mathscr{O},\mathfrak{w})\},$$

where

$$\|u\|^2_{H^1(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} y\left(u_x^2 + u_y^2\right) \mathfrak{w}\, dxdy + \int_{\mathscr{O}} (1+y) u^2\, \mathfrak{w}\, dxdy.$$

Weighted Sobolev spaces (continued)

Let $H^1_0(\mathscr{O},\mathfrak{w})$ be the closure in $H^1(\mathscr{O},\mathfrak{w})$ of $C^1_c(\mathscr{O}) \cap H^1(\mathscr{O},\mathfrak{w})$. For $i=0,1$, let $H^1_0(\mathscr{O}\cup\Gamma_i,\mathfrak{w})$ be the closure in $H^1(\mathscr{O},\mathfrak{w})$ of $C^1_c(\mathscr{O}\cup\Gamma_i) \cap H^1(\mathscr{O},\mathfrak{w})$, where

$$\Gamma_0 = (x_0,x_1)\times\{0\} \text{ and } \Gamma_1 = \{x_0,x_1\}\times(0,\infty),$$

and $\Gamma_1 = \{x_0\}\times(0,\infty)$ if $x_1=+\infty$, $\Gamma_1=\{x_1\}\times(0,\infty)$ if $x_0=-\infty$, and $\Gamma_1=\emptyset$ if $x_0=-\infty$ and $x_1=+\infty$.

Gårding inequality

Proposition

Let $q, r, \sigma, \kappa, \theta \in \mathbb{R}$ *be constants such that*

$$\beta := \frac{2\kappa\theta}{\sigma^2} > 0, \quad \sigma \neq 0, \quad \text{and} \quad -1 < \rho < 1.$$

Then for all $u \in V$ *such that* $u = 0$ *on* Γ_1, *where* $V = H^1(\mathscr{O}, \mathfrak{w})$,

$$a(u, u) \geq \frac{1}{2} C_2 \|u\|_V^2 - C_3 \|(1+y)^{1/2} u\|_{L^2(\mathscr{O}, \mathfrak{w})}^2.$$

Continuity estimates

Proposition

Choose

- $\beta < 1$: $V = H^1(\mathscr{O}, \mathfrak{w})$ *and* $W = H^1_0(\mathscr{O} \cup \Gamma_1, \mathfrak{w})$;
- $\beta > 1$: $V = W = H^1(\mathscr{O}, \mathfrak{w})$.

Then

$$|a(u,v)| \leq C_1 \|u\|_V \|v\|_W, \quad \forall (u,v) \in V \times W,$$

where C_1 *is a positive constant depending at most on the coefficients* $r, q, \kappa, \theta, \rho, \sigma$.

Elliptic variational inequality with (nonhomogeneous) Dirichlet boundary conditions

Let $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $g, \psi \in H^1(\mathscr{O}, \mathfrak{w})$ such that $\psi \leq g$ on $\mathscr{O}$. For $\beta > 1$, find $u \in H^1(\mathscr{O}, \mathfrak{w})$ such that

$$a(u, v-u) \geq (f, v-u)_{L^2(\mathscr{O},\mathfrak{w})}, \text{ with } u \geq \psi \text{ on } \mathscr{O} \text{ and } u = g \text{ on } \Gamma_1,$$
$$\forall v \in H^1(\mathscr{O}, \mathfrak{w}) \text{ with } v \geq \psi \text{ on } \mathscr{O} \text{ and } v = g \text{ on } \Gamma_1,$$

that is, $u - g, v - g \in H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w})$. For $\beta < 1$, the statement is identical, except that the Dirichlet conditions are $u = g$ and $v = g$ on Γ, that is, $u - g, v - g \in H^1_0(\mathscr{O}, \mathfrak{w})$.

Existence and uniqueness of solutions to the elliptic variational inequality

Theorem

There exists a unique solution to the elliptic variational inequality for the Heston operator.

Higher order regularity

Definition

Let

$$H^2(\mathscr{O},\mathfrak{w}) := \{u \in L^2(\mathscr{O},\mathfrak{w}) : (1+y)^{1/2}u, y^{1/2}Du, yD^2u \in L^2(\mathscr{O},\mathfrak{w})\},$$

where

$$\|u\|^2_{H^2(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left[y^2\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right) \right. \\ \left. + y\left(u_x^2 + u_y^2\right) + (1+y)u^2\right] \mathfrak{w}\, dxdy.$$

Let $H^2_{\mathrm{loc}}(\mathscr{O},\mathfrak{w})$ denote the space of functions $u \in H^2(\mathscr{O}',\mathfrak{w})$ for all $\mathscr{O}' \Subset \mathscr{O}$.

H^2 regularity for solutions to the elliptic Heston variational inequality

Theorem

Suppose $\psi(x,y) = (K - e^x)^+$ or $(e^x - K)^+$. If u is the solution to the Heston elliptic variational inequality, then $u \in H^2(\mathscr{O}, \mathfrak{w})$.

Strong formulation of the elliptic variational inequality

If $u \in H^2(\mathscr{O}, \mathfrak{w})$ and $\psi \in H^1(\mathscr{O}, \mathfrak{w})$, then the variational formulation has an equivalent *strong formulation* as a complementarity problem, which is to find $u \in V$ such that

$$Au - f \geq 0, \quad u - \psi \geq 0, \quad (Au - f)(u - \psi) = 0 \text{ on } \mathscr{O}.$$

Parabolic or evolutionary variational inequalities for the Heston operator

- Simple attempts to adapt the argument Bensoussan and Lions (1982) in their proof existence and uniqueness of solutions to the "strong" variational inequality to the Heston operator A fail because the bilinear map defined by A is *non-coercive*.

A change of dependent variable

- To circumvent the lack of coerciveness, we employ the change of dependent variable

$$\tilde{u}(t,x,y) = e^{-\lambda(1+y)(T-t)}u(t,x,y), \quad u \in V, (t,x,y) \in Q,$$

by analogy with the familiar *exponential shift* change of dependent variable $\tilde{u} = e^{-\lambda(T-t)}u$.

A change of dependent variable (continued)

- One finds that the *non-coercive* parabolic problem,

$$-u' + Au = f \text{ on } Q, \quad u(T) = h \text{ on } \mathscr{O}, \quad u = g \text{ on } \Sigma,$$

  is transformed, for $t \in [T-\delta, T]$ and sufficiently small δ, into an equivalent *coercive* parabolic problem,

$$-\tilde{u}' + \tilde{A}\tilde{u} = \tilde{f} \text{ on } Q, \quad \tilde{u}(T) = h \text{ on } \mathscr{O}, \quad \tilde{u} = \tilde{g} \text{ on } \Sigma,$$

- An obstacle condition $u \geq \psi$ is transformed into an equivalent obstacle condition $\tilde{u} \geq \tilde{\psi}$.

A change of dependent variable (continued)

The bilinear form on $V \times V$ (defined by the weight $\mathfrak{w}$) associated to the operator $\tilde{A}(t)$ (with suitable boundary conditions) is

$$\tilde{a}(t; \tilde{u}(t), v) := (\tilde{A}(t)\tilde{u}(t), v)_{L^2(\mathscr{O},\mathfrak{w})}. \tag{1}$$

We then obtain the key continuity estimate and Gårding inequality for $\tilde{a}(t)$.

Continuity estimate and Gårding inequality for the transformed Heston operator

Proposition

For a sufficiently large positive constant λ, depending only the coefficients of A, and a sufficiently small positive constant $\delta < T$, depending only on λ and the coefficients of A, the bilinear map $\tilde{a}(t) : V \times V \to \mathbb{R}$ obeys

$$
\begin{aligned}
|\tilde{a}(t; u, v)| &\leq C\|u\|_V \|v\|_V, \\
\tilde{a}(t; v, v) &\geq \frac{\alpha}{2}\|v\|_V^2,
\end{aligned}
$$

for all $u, v \in V$ and $t \in [T - \delta, T]$.

Change of Sobolev weight and transformation back to original problem

The weight in our previous definition of weighted Sobolev spaces,

$$\mathfrak{w}(x,y) := \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x|-\mu y}, \quad (x,y) \in \mathscr{O},$$

is replaced, when transforming back from a solution $\tilde{u}$ to a solution u to the original problem, by

$$\begin{aligned}\tilde{\mathfrak{w}}(x,y) &:= e^{-2\lambda M(1+y)} \mathfrak{w}(x,y) \\ &= \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x|-\mu y - 2\delta\lambda(1+y)}, \quad (x,y) \in \mathscr{O},\end{aligned}$$

where $M > T$ is a constant.

Setup for abstract parabolic equations and inequalities

Let V be a reflexive Banach space with dual V'. Denote $\mathscr{V} = L^2(0, T; V)$, with dual $\mathscr{V}' = L^2(0, T; V')$. Let H be a Hilbert space. The embeddings

$$V \hookrightarrow H \cong H' \hookrightarrow V',$$

are continuous, with $V \subset H$ dense. Let $\mathscr{A} : \mathscr{V} \to \mathscr{V}'$ be a continuous but not necessarily linear map. Typically, $\mathscr{A}(t, v) = \mathscr{A}(t)v(t)$, where $\mathscr{A}(t) : V \to V'$, $t \in [0, T]$. When $\mathscr{A}(t) \in \mathscr{L}(V, V')$, the transformed bilinear form $a(t) : V \times V \to \mathbb{R}$ is

$$a(t; u, v) := \mathscr{A}(t)u(v), \quad u, v \in V.$$

If $u \in D(\mathscr{A}(t)) = \{v \in V : \mathscr{A}(t)v \in H\}$, we write

$$(A(t)u, v)_H := \mathscr{A}(t)u(v), \quad v \in V.$$

Formulations of the Cauchy problem

Proposition (Showalter)

Let $u_0 \in H$ and $f \in \mathscr{V}'$. If $u \in \mathscr{V}$, the following are equivalent.

1. *(Strong)*

$$-u' + \mathscr{A} u = f \text{ in } \mathscr{V}', \quad u(T) = u_0.$$

2. *(Variational) For each $v \in \mathscr{V} \cap W^{1,2}(0,T;H)$ with $v(0) = 0$,*

$$\int_0^T \left[(u, v') + \mathscr{A} u(v) - f(v)\right] \, dt - (u_0, v(T))_H = 0.$$

3. *(Weak) For each $v \in V$,*

$$-\frac{d}{dt}(u,v)_H + \mathscr{A} u(v) = f(v) \text{ in } \mathscr{D}^*(0,T), \quad u(T) = u_0.$$

Existence and uniqueness for the abstract linear Cauchy problem

Proposition (Showalter)

Assume the operators $\mathscr{A}(t)$ are in $\mathscr{L}(V, V')$ and that there is a constant $\alpha > 0$ such that

$$\mathscr{A}(t)v(v) \geq \alpha \|v\|_V^2, \quad v \in V, \quad t \in [0, T].$$

Given $f \in \mathscr{V}'$, $u_0 \in H$, there is a unique solution $u \in \mathscr{V}$, $u' \in \mathscr{V}'$ to

$$-u' + \mathscr{A}u = f \text{ in } \mathscr{V}', \quad u(T) = u_0.$$

and u satisfies

$$\|u\|_{\mathscr{V}}^2 \leq (1/\alpha)^2 \left(\|f\|_{\mathscr{V}'}^2 + \|u_0\|_H^2\right).$$

Variational inequalities and the penalization method

- One may use the penalization method as a stepping stone from existence (and regularity) for non-linear elliptic and parabolic equations to existence and regularity for solutions to elliptic and parabolic variational inequalities.
- Existence and uniqueness for the Cauchy problem for the penalized parabolic equation follows from existence and uniqueness results for the non-linear abstract Cauchy problem (Showalter, 1997).

Existence and uniqueness for the abstract Cauchy problem for the *penalized* equation

Proposition

Let $\mathscr{A}(t,\cdot) \in \mathscr{L}(V, V')$, $t \in [0,T]$ obey

1. *The function $\mathscr{A}(\cdot, v) : [0,T] \to V'$ is measurable, $\forall v \in V$.*
2. *There is a positive constant α such that*

$$\mathscr{A}(t,v)(v) \geq \alpha \|v\|_V^2, \quad t \in [0,T], v \in V.$$

Then, given $\psi \in \mathscr{H}$, $f \in \mathscr{V}'$, $u_0 \in H$ with $u_0 \geq \psi(T,\cdot)$, and $\varepsilon > 0$, there is a unique solution, $u_\varepsilon \in \mathscr{V}$, with $u_\varepsilon' \in \mathscr{V}'$, to

$$-u_\varepsilon' + \mathscr{A} u_\varepsilon + \frac{1}{\varepsilon}(\psi - u_\varepsilon)^+ = f \text{ in } \mathscr{V}', \quad u_\varepsilon(T) = u_0 \text{ in } H.$$

Strong problem for a parabolic variational inequality

Let $\mathscr{K} \subset \mathscr{V}$ be a convex subset. Given $f \in \mathscr{V}'$ and $u_0 \in H$, $u \in \mathscr{V}$ solves the *strong problem* if

$$u \in \mathscr{K}, u' \in \mathscr{V}'$$

$$-\int_0^T u'(v-u)\,dt + \mathscr{A}u(v-u) \geq f(v-u), \quad \forall v \in \mathscr{K},$$

$$u(T) = u_0.$$

Weak problem for a parabolic variational inequality

Given $f \in \mathscr{V}'$ and $u_0 \in H$, $u \in \mathscr{V}$ solves the *weak problem* if

$$u \in \mathscr{K},$$

$$-\int_0^T v'(v-u)\,dt + \mathscr{A}u(v-u) \geq f(v-u),$$

$$\forall v \in \mathscr{K} \text{ with } v' \in \mathscr{V}', v(T) = u_0.$$

Existence and uniqueness for the *weak* problem for a parabolic variational inequality

Suppose $K(t)$, $t \in [0, T]$, is a non-decreasing family of closed, convex subsets of V containing $u_0 \in H$. Then

$$\mathscr{K} = \{v \in \mathscr{V} : v(t) \in K(t) \text{ a.e. } t \in [0, T]\}$$

is a closed and convex subset of $\mathscr{V}$. The next theorem is an application of results of Showalter on abstract parabolic variational inequalities in Banach spaces.

Existence and uniqueness for the *weak* problem for a parabolic variational inequality (continued)

Theorem (Showalter)

Suppose $\mathscr{A}(t,\cdot) \in \mathscr{L}(V, V')$ are given with $\mathscr{A}(t,v)$ measurable in $t \in [0,T]$, $\forall v \in V$, and

$$\mathscr{A}(t,v)(v) \geq \alpha \|v\|_V^2, \quad \forall v \in V, t \in [0,T],$$

for some $\alpha > 0$. Suppose $K(t)$, $t \in [0,T]$, is a non-decreasing family of closed, convex subsets of V containing $u_0 \in H$. Then for each $f \in \mathscr{V}'$ there is a unique solution $u \in \mathscr{K}$ to

$$\int_0^T (-v' + \mathscr{A}u - f)(v-u)\,dt \geq 0,$$

$$\forall v \in \mathscr{K} \text{ with } v' \in \mathscr{V}', v(T) = u_0.$$

Existence and uniqueness for the *strong* problem for a parabolic variational inequality

- Ultimately, we want a classical solution to the familiar "complementarity" formulation of the American-style option pricing problem.
- We can obtain such classical solutions by developing a regularity theory for solutions to the weak problem.
- It is more direct to adapt the Bensoussan-Lions approach using the Galerkin and penalization methods to establish existence and uniqueness for the strong problem for a parabolic variational inequality.

Existence and uniqueness for the *strong* problem for a parabolic variational inequality (continued)

Theorem

Suppose $\mathscr{A}(t,\cdot) \in \mathscr{L}(V, V')$ *are given with* $\mathscr{A}(t,v)$ *measurable in* $t \in [0,T]$, $\forall v \in V$, *and, for some* $\alpha > 0$,

$$\mathscr{A}(t,v)(v) \geq \alpha \|v\|_V^2, \quad \forall v \in V, t \in [0,T],$$

Let $\psi \in W^{1,2}(0,T;H)$, $\mathscr{K} = \{v \in \mathscr{V} : v \geq \psi\}$, $u_0 \in \mathscr{K}$, *and* $f \in \mathscr{H}$. *Then there is a unique solution* $u \in \mathscr{K}$, $u' \in \mathscr{H}$ *to*

$$\int_0^T (-u' + \mathscr{A}u - f)(v - u)\, dt \geq 0, \quad \forall v \in \mathscr{K}, \quad u(T) = u_0.$$

Regularity for solutions to the strong problem for the parabolic *Heston* variational inequality

Using our weighted Sobolev spaces and estimates, we adapt the Bensoussan-Lions regularity theory to establish

Theorem

In the situation of the existence and uniqueness theorem for the strong problem for a parabolic Heston variational inequality, suppose $\psi(t,x,y) = (e^x - K)^+$ *or* $(K - e^x)^+$. *Then the solution* u *is in* $L^2(0,T;H^2(\mathscr{O},\mathfrak{w}))$.

Given this regularity, a solution to the strong problem for the parabolic Heston variational inequality is a solution to the more familiar complementarity form for the Heston variational inequality:

Complementarity form of the Heston variational inequality

Theorem

Given $f \in L^2(0,T; L^2(Q,\mathfrak{w}))$, $g \in L^2(0,T; H^2(\mathscr{O},\mathfrak{w}))$, $u_0(x,y) = \psi(t,x,y) = (e^x - K)^+$ *or* $(K - e^x)^+$, *then there is a unique* $u \in L^2(0,T; H^2(\mathscr{O},\mathfrak{w}))$ *solving*

$$
\begin{aligned}
-u' + Au &\geq f \text{ on } Q,\\
u &\geq \psi \text{ on } Q,\\
(-u' + Au - f)(u - \psi) &= 0 \text{ on } Q,\\
u &= g \text{ on } (\Sigma_0 \cup \Sigma_1) \times [0,T) \text{ (if } 0 < \beta < 1\text{) or}\\
&= g \text{ on } \Sigma_1 \times [0,T) \text{ (if } \beta \geq 1\text{)},\\
u(T) &= \psi \text{ on } \mathscr{O}.
\end{aligned}
$$

where $g \geq \psi$ *on* Σ.

Current research

- Global $W^{2,p}$ regularity.
- Regularity of the solution u up to the boundary.
- Regularity of the free boundary separating the continuation and exercise regions.

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