Pricing Options on Realized Variance in Lévy Models

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based on joint work with Johannes Muhle-Karbe

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Realized Variance

Let S denote a discounted asset, and X its logarithm.

Realized Variance

The **annualized realized variance** of X over the period $[0, T]$ subdivided into n business days $0 = t_0 < \cdots < t_n = T$ is given by

$$RV_n(T) = \frac{1}{T} \sum_{k=1}^{n} \left(\log \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 = \frac{1}{T} \sum_{k=1}^{n} \left(X_{t_k} - X_{t_{k-1}} \right)^2$$

- A considerable number of financial instruments use realized variance as an underlying: **variance swap, volatility swap, calls/puts on realized variance**

Approximation by Quadratic Variation (1)

- Standard pricing approach: substitute annualized *quadratic variation* $QV(T) = \frac{1}{T}[X,X]_T$ for *realized variance*.

$$RV_n(T) \approx QV(T)$$

- Quadratic variation is the limit in probability of realized variance, when T stays fixed and the number of increments n tends to infinity.

Approximation by Quadratic Variation (2)

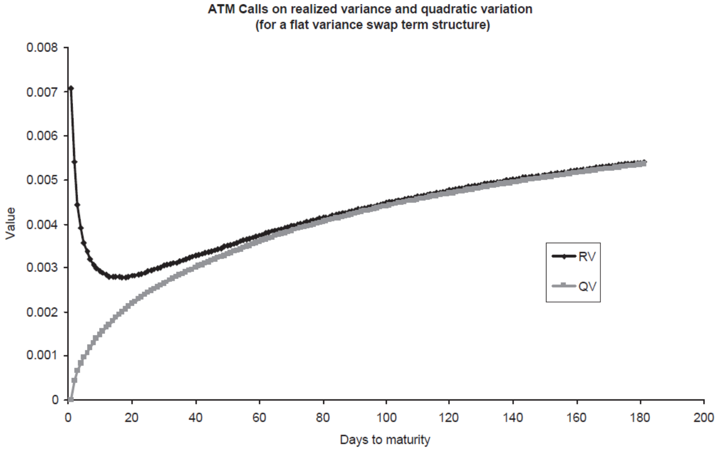
- The approximation via quadratic variation works well for claims with (approximately) linear payoffs: variance swaps, volatility swaps.

  See Bühler [2006], Sepp [2008], Broadie and Jain [2008]

- The approximation is not sufficient for claims with non-linear payoffs like calls/puts and for maturities shorter than 3 months.

  See Bühler [2006], Gatheral [2008].

Bühler's Example



ATM call in the Heston model. Plot taken from Bühler [2006].

Main Questions

This talk addresses the following questions:

1. How big is the *discretization gap* between options on quadratic variation (QV) and realized variance (RV)?
2. How can options on the realized variance be valuated exactly?

We focus on ATM calls, i.e. options with payoff

$$(RV_n(T) - \mathbb{E}[RV_n(T)])^+$$

where $\mathbb{E}[RV_n(T)]$ is the swap rate.

The Discretization Gap

As a proxy for the short-time behavior of options on **realized variance** we use

$$\lim_{T \to 0} \mathbb{E}\left[(RV_1(T) - \mathbb{E}[RV_1(T)])^+\right];$$

for options on **quadratic variation** we use

$$\lim_{T \to 0} \mathbb{E}\left[(QV(T) - \mathbb{E}[QV(T)])^+\right].$$

The discretization gap is the difference between the two:

$$\lim_{T \to 0} \left\{\mathbb{E}\left[(RV_1(T) - \mathbb{E}[RV_1(T)])^+\right] - \mathbb{E}\left[(QV(T) - \mathbb{E}[QV(T)])^+\right]\right\}.$$

Note that $RV_1(T)$ is the realized variance over a single business day, i.e. $RV_1(T) = \frac{1}{T} X_T^2$

Underlying: Lévy process

We assume that the underlying X follows a Lévy process:

X can be characterized by its Lévy triplet (b, σ^2, F), or by its *Lévy exponent*

$$\psi(u) = bu + \frac{\sigma^2}{2}u^2 + \int (e^{ux} - 1 - uh(x))F(dx).$$

We also assume that the first two moments of X exist. In this case X has a decomposition

$$X_t = bt + \sigma W_t + L_t$$

where L is a centered pure-jump process of finite variance, and W an independent Brownian motion.

Results for Lévy processes – Quadratic variation

Theorem (K.-R. and Muhle-Karbe (2010))

For a Lévy process X a call on **quadratic variation** *satisfies*

$$\lim_{T \to 0} \mathbb{E}\left[(QV(T) - \mathbb{E}[QV(T)])^+\right] = v^2,$$

where $v^2 = \int x^2 F(dx)$.

Note: v^2 is the variance of the pure jump component L.

Results for Lévy processes – Realized variance

Theorem (K.-R. and Muhle-Karbe (2010))

For a Lévy process X a call on **realized variance** *satisfies*

$$\lim_{T \to 0} \mathbb{E}\left[(RV_1(T) - \mathbb{E}[RV_1(T)])^+\right] = \sigma^2 P\left(\frac{v^2}{\sigma^2}\right) + v^2 Q\left(\frac{v^2}{\sigma^2}\right),$$

where $v^2 = \int x^2 F(dx)$ and $P(r)$ resp. $Q(r)$ are strictly decreasing resp. increasing functions on $[0, \infty)$, given by

$$P(r) = \sqrt{\frac{2(1+r)}{\pi \exp(1+r)}}, \quad \text{and} \quad Q(r) = 2\Phi(\sqrt{1+r}) - 1,$$

with $\Phi(.)$ denoting the standard normal distribution function.

Special cases – Pure diffusion

Pure diffusion – no jumps:

$$\lim_{T\to 0} \mathbb{E}\left[(QV(T) - \mathbb{E}[QV(T)])^+\right] = 0$$

$$\lim_{T\to 0} \mathbb{E}\left[(RV_1(T) - \mathbb{E}[RV_1(T)])^+\right] = \sqrt{\frac{2}{\pi e}}\sigma^2 \approx 0.48\sigma^2$$

Under mild conditions these results also hold in pure-diffusion models with stochastic volatility (but without leverage effect).

Special cases – Pure jump

Pure jump process – no diffusion:

$$\lim_{T \to 0} \mathbb{E}\left[(QV(T) - \mathbb{E}[QV(T)])^+\right] = v^2$$
$$\lim_{T \to 0} \mathbb{E}\left[(RV_1(T) - \mathbb{E}[RV_1(T)])^+\right] = v^2$$

The discretization gap vanishes completely in pure-jump models!

In **true jump-diffusion** models the interaction between jump and diffusion component is surprisingly complex.

Numerical Confirmation

Numerical results for 2 Lévy-based models with 3 different parameter sets:

- The Kou model is a jump-diffusion model with double-exponentially distributed jump sizes.
- The CGMY model is a pure jump model introduced by Carr, Geman, Madan and Yor.
- We use calibrated parameter sets from Sepp [2008] and Carr et al. [2005] respectively. For the Kou model we also look at the effect of reducing the diffusion volatility σ from 0.3 to 0.2.

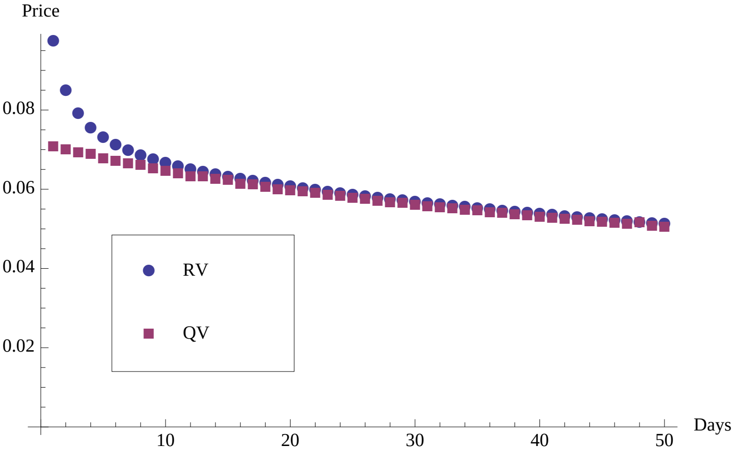


Figure: ATM call prices on normalized quadratic variation resp. realized variance in the Kou model for $\sigma = 0.3$. The analytic short-time limits from the corresponding theorems are 0.0718 resp. 0.0980.

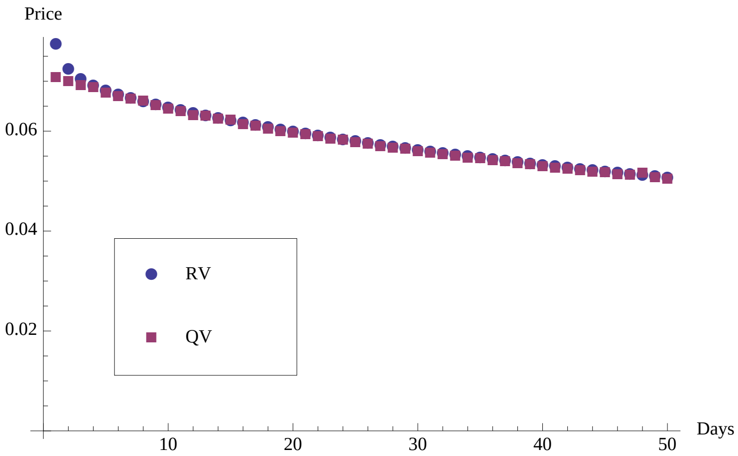


Figure: ATM call prices on normalized quadratic variation resp. realized variance in the Kou model for $\sigma = 0.2$. The analytic short-time limits from the corresponding theorems are 0.0706 resp. 0.0773.

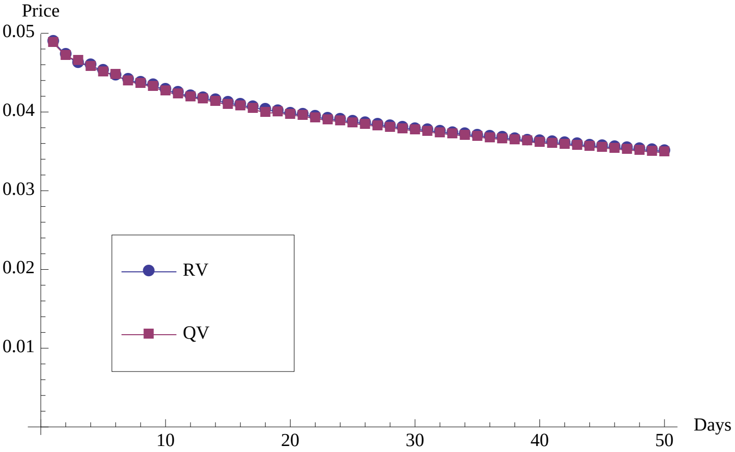


Figure: ATM call prices on normalized quadratic variation resp. realized variance in the CGMY model. The discretization gap vanishes as predicted. Go to generalized result

Numerical Methods

How did we produce the numerical results?

- Monte-Carlo simulation can be problematic for Lévy processes: transition density not known in closed form, jumps may have infinite arrival rate, etc. . .
- For quadratic variation Fourier-based methods have been described in the previous talk.
- For realized variance we propose analogous methods in [K.-R. and Muhle-Karbe (2010)].

Fourier Pricing

Suppose the Laplace transform $\mathbb{E}\left[e^{-uX_t^2}\right]$ of the squared Lévy process is known in the half plane $\mathcal{H}_+ = \{u \in \mathbb{C} : \mathrm{Re}(u) \geq 0\}$.

Applying the Fourier-pricing approach of Carr & Madan yields:

Fourier Pricing for calls on realized variance

$$\mathbb{E}\left[(RV_n(T) - K)^+\right] =$$
$$= \mathbb{E}\left[RV_n(T)\right] - K + \frac{1}{\pi}\int_{\alpha}^{\alpha+i\infty} \mathrm{Re}\left(\frac{e^{Ku}}{u^2}\mathbb{E}\left[\exp\left(-uX_\delta^2\right)\right]^n\right) du$$

where $\alpha > 0$ and $\delta = T/n$.

Fourier Pricing: The Laplace transform of X_t^2

Theorem (K.-R. and Muhle-Karbe (2010))

Let X_t be a Lévy process, whose characteristic exponent $\psi(u)$ satisfies a mild analyticity condition. Let Z be an independent standard normal random variable. Then

$$\mathbb{E}\left[e^{-uX_t^2}\right] = \mathbb{E}\left[e^{t\psi(iZ\sqrt{2u})}\right]$$

holds for all u in the complex half-plane $\mathcal{H}_+ = \{u : \mathrm{Re}(u) > 0\}$.

- Replaces the integration with respect to the law of the Lévy process by an integration with respect to a normal distribution.
- The analyticity condition holds e.g. for the Kou and the Merton model, the NIG, the Variance Gamma and the CGMY process.

Summary

- In many cases quadratic variation is not a good proxy for realized variance, when pricing of call/put options on realized variance is concerned.
- The difference in prices is most pronounced in diffusion models, decreases when jumps are added, and vanishes completely in pure-jump models.
- We have presented methods for exact valuation of options on realized variance by Fourier methods in the context of exponential-Lévy models.
- Extensions to stochastic volatility models with jumps are work in progress.

Thank you for your attention!

For details see:

Keller-Ressel, M. and Muhle-Karbe, J. (2010). *Asymptotics and exact pricing of options on variance*. arXiv:1003.5514.

M. Broadie and A. Jain. The effect of jumps and discrete sampling on volatility and variance swaps. *International Journal of Theoretical and Applied Finance*, 11:761–797, 2008.

Hans Bühler. *Volatility Markets – Consistent modeling, hedging and practical implementation*. PhD thesis, TU Berlin, 2006.

P. Carr, H. Geman, D. Madan, and M. Yor. Pricing options on realized variance. *Finance and Stochastics*, 9:453–475, 2005.

Jim Gatheral. Consistent modeling of SPX and VIX options. Presentation at the 5th Bachelier Congress, London, 2008.

A. Sepp. Analytical pricing of double-barrier options under a double-exponential jump-diffusion process. *International Journal of Theoretical and Applied Finance*, 7:151–175, 2008.

Generalization of the realized variance result

Theorem (Generalized short-time limit)

For a Lévy process X a call on **realized variance** *satisfies*

$$\lim_{T\to 0} \mathbb{E}\left[\left(RV_n(T) - k\mathbb{E}\left[RV_n(T)\right]\right)^+\right] =$$
$$\sigma^2 P_{k,n}\left(\frac{v^2}{\sigma^2}\right) + \left(\sigma^2(k-1) + v^2 k\right) Q_{k,n}\left(\frac{v^2}{\sigma^2}\right),$$

where $v^2 = \int x^2 F(dx)$ *and* $P_{k,n}(r)$ *and* $Q_{k,n}(r)$ *are given by*

$$P_{k,n}(r) = \frac{2/n}{\Gamma(n/2)} \left(\frac{nk(1+r)}{2\exp(k(1+r))}\right)^{n/2}$$
$$Q_{k,n}(r) = \gamma_0(n/2, nk(1+r)/2)$$

with $\gamma_0(.,.)$ *denoting the regularized incomplete Gamma function.*

Back to Numerical Results