Optimal Liquidation of an Indivisible Asset with Independant Investment

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We consider the point of view of an agent who possesses

- a portfolio of assets
- an indivisible asset : small family firm, piece of land, factory etc...

He wants to maximize his total wealth at the sell time of the indivisible asset.

$\Longrightarrow$ This was introduced by Henderson and Hobson :

"An explicit solution for an optimal stopping/optimal control problem which models an asset sale", *The annals of Applied Probability*, 2008.

We consider the problem $w(x) = \sup_X \mathbb{E}\big[G(X_T)\big]$.

where X is a martingale, G is a concave value function and $T > 0$.

- By optimality : $w(x) \geq G(x)$
- By Jensen's inequality : $w(x) \leq \sup_X \mathbb{E}\big[G(X_T)\big] \leq U(x)$

Then $w(x) = G(x)$ and the optimal strategy is to keep the wealth constant.

Outline

- The mixed investment/sale problem
- Dynamic Programming Equation in a continuous framework
- Determination of the Value function
- The ε-optimal strategies
- An existence result
- Conclusion

We consider $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space. Let B be a $\mathcal{F}_t$ Brownian motion valued in $\mathbb{R}$.

Let Y be the price process of one unit of an indivisible asset modelled by

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t \qquad Y_0 = y > 0$$

Moreover, we assume :

$$\mu(0) > 0 \text{ and } \sigma(0) = 0$$

We consider a concave function U from $\mathbb{R}^+$ to $\mathbb{R}^+$.

We want to solve the following problem

$$V(x,y) = \sup_{\substack{X \in \mathcal{M}^{\perp}(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}[U(X_\tau + Y_\tau^y)]$$

where

(i) $(x,y) \in D = \{\mathbb{R} \times \mathbb{R}_*^+;\ x+y \geq 0\}$.

(ii) $\mathcal{M}^{\perp}(x,y) = \Big\{X$ càdlàg martingale such that for all $t \geq 0$

$\mathbb{E}[X_t] = x;\ [X, Y^y]_t = 0;\ X_t + Y_t^y \geq 0\Big\}$.

(iii) $\tau \in \mathcal{T}$ where $\mathcal{T}$ is the set of all stopping times adapted to $\mathbb{F}$.

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Dynamic Programming Equation in a continuous framework

$$V^0(x,y) := \sup_{\substack{\alpha \in \mathcal{A}^\perp(Y) \\ \tau \in \mathcal{T}}} \mathbb{E}\left[U\left(x + \int_0^\tau \alpha_u \, dW_u + Y_\tau^y\right)\right]$$

where

- W is an $\mathcal{F}_t$ Brownian motion valued in $\mathbb{R}$ such that $\langle W, B\rangle_t = 0$.
- $\mathcal{A}^\perp(Y)$ is the "continuous version" of $\mathcal{M}^\perp(Y)$.

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Define the lower semicontinuous hull of V^0 by

$$V^0_*(x,y) = \liminf_{\substack{x'\to x\\ y'\to y}} V^0(x,y)$$

Proposition

Assume that V^0 is locally bounded, then V^0_* is a viscosity supersolution of

$$\min\{-\frac{1}{2}\sigma(y)^2 v_{yy} - \mu(y)v_y; -v_{xx}; v - U(x+y)\} = 0 \text{ on } D$$

We assume that :

$$\forall x \in \mathbb{R}_*^+ \ \sigma^2(y) > 0 \text{ and } \frac{|\mu(y)|}{\sigma^2(y)} \in \mathbb{L}^1_{loc}(\mathbb{R}_*^+)$$

We consider the process Z defined by $Z := S\left(Y^y\right)$ where S is the solution of

$$\mu(y)S'(y) + \frac{1}{2}\sigma^2(y)S''(y) = 0$$

$\Longrightarrow$ S is correctly defined and Z is a local martingale.

Let us introduce $D' = \left\{(x,z) \in \mathbb{R}^2 \ : \ x + S^{-1}(z) \geq 0\right\}$.

Define $\bar{V}^0(x,z) := V^0\left(x, S^{-1}(z)\right)$ and $\bar{V}^0_*$ its associated upper semicontinuous hull on D'.

We define $\bar{U}(x,z) := U\left(x + S^{-1}(z)\right)$.

Proposition

Assume that $\bar{V}^0$ is locally bounded. Then $\bar{V}^0_*$ is a viscosity supersolution of

$$\min\left\{ -\bar{v}_{yy} \ ; \ -\bar{v}_{xx} \ ; \ \bar{v} - \bar{U} \right\} = 0 \text{ on } D'$$

- We define $\bar{U}^{\infty}$ by $\bar{U}^{\infty} = \lim_n \bar{U}_n$ where $\left(\bar{U}_n\right)_n$ is such that

$$\bar{U}_0 = \bar{U}$$
$$\bar{U}_{2n} = (\bar{U}_{2n-1})^{conc_x}$$
$$\bar{U}_{2n+1} = (\bar{U}_{2n})^{conc_y}$$

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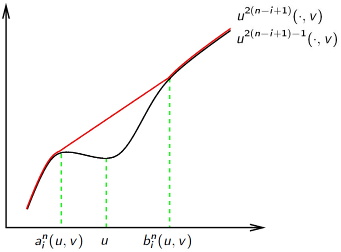
- Thanks to convolution arguments, we can regularize $\bar{U}^\infty$. Applying Itô's formula, we get that $\bar{U}^\infty(X_t, Z_t)$ is a positive supermartingale and then :

$$
V(x, y) \leq \sup_{\substack{X \in \mathcal{M}^\perp(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}[\bar{U}^\infty(X_\tau, Z_\tau)] \leq \bar{U}^\infty(x, S(y))
$$

Then for all $(x, y) \in D$, $V(x, y) = \bar{U}^\infty(x, S(y))$.

- For two given random variables (u, v),



- We define the random variable η_i^n by :

$$\eta_i^n(u, v) = \begin{cases} a_i^n(u, v) & \text{with proba } p_i^n(u, v) \\ b_i^n(u, v) & \text{with proba } 1 - p_i^n(u, v) \end{cases}$$

and $u = p_i^n(u, v) a_i^n(u, v) + \left(1 - p_i^n(u, v)\right) b_i^n(u, v)$.

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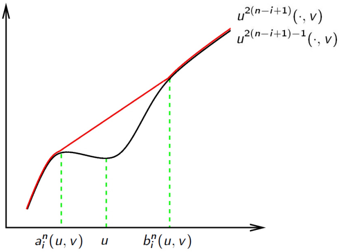

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- We define the pure jump martingale X^n as follows :

$$X_t^n = x \quad \forall t \in [0, \tau_1^n[$$
$$X_t^n = \eta_1^n(X_{\tau_0^n}^n, Z_{\tau_1^n}) \quad \forall t \in [\tau_1^n, \tau_2^n[$$
$$\vdots$$
$$X_t^n = \eta_i^n(X_{\tau_{i-1}^n}^n, Z_{\tau_i^n}) \quad \forall t \in [\tau_i^n, \tau_{i+1}^n[$$

$\Longrightarrow$ Optimal investment problem with fixed random maturity and non concave utility function

- We define the sequence of stopping times $(\tau^n)_{n\geq 0}$ for $i \in \{0 \dots n+1\}$ by

$$\tau_0^n = \inf\{t \geq 0 : \bar{U}^\infty(x, Z_t) = \bar{U}^{2n+1}(x, Z_t)\}$$
$$\tau_i^n = \inf\{t \geq \tau_{i-1}^n : \bar{U}^{2(n-i+1)+1}(X_{\tau_{i-1}^n}^n, Z_t) = \bar{U}^{2(n-i+1)}(X_{\tau_{i-1}^n}^n, Z_t)\}$$
$$\vdots$$
$$\tau_{n+1}^n = \inf\{t \geq \tau_n^n : \bar{U}^1(X_{\tau_n^n}^n, Z_t) = \bar{U}^0(X_{\tau_n^n}^n, Z_t)\}$$

$\Longrightarrow$ Optimal stopping problem with fixed investment

- We define the pure jump martingale X^n as follows :

$$X^n_t = x \quad \forall t \in [0, \tau^n_1[$$
$$X^n_t = \eta^n_1(X^n_{\tau^n_0}, Z_{\tau^n_1}) \quad \forall t \in [\tau^n_1, \tau^n_2[$$
$$\vdots$$
$$X^n_t = \eta^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}) \quad \forall t \in [\tau^n_i, \tau^n_{i+1}[$$

$\Longrightarrow$ Optimal investment problem with fixed random maturity and non concave utility function

- We define the sequence of stopping times $(\tau^n)_{n\geq 0}$ for $i \in \{0 \ldots n+1\}$ by

$$\tau^n_0 = \inf\left\{t \geq 0 : \bar{U}^{\infty}(x, Z_t) = \bar{U}^{2n+1}(x, Z_t)\right\}$$
$$\tau^n_i = \inf\left\{t \geq \tau^n_{i-1} : \bar{U}^{2(n-i+1)+1}(X^n_{\tau^n_{i-1}}, Z_t) = \bar{U}^{2(n-i+1)}(X^n_{\tau^n_{i-1}}, Z_t)\right\}$$
$$\vdots$$
$$\tau^n_{n+1} = \inf\left\{t \geq \tau^n_n : \bar{U}^{1}(X^n_{\tau^n_n}, Z_t) = \bar{U}^{0}(X^n_{\tau^n_n}, Z_t)\right\}$$

$\Longrightarrow$ Optimal stopping problem with fixed investment

Proposition

Assume that $\exists K$ compact subset of D' such that
$\forall (x,z) \notin K \quad \bar{U}^{\infty}(x,z) = \bar{U}(x,z)$

Then for all (x,y) in D, for any positive constant ε, there exists n such that

$$\varepsilon + \mathbb{E}\left[\bar{U}^0(X^n_{\tau^n_{n+1}}, X_{\tau^n_{n+1}})\right] \geq \bar{U}^{\infty}(x, S(y))$$

and

$$\bar{U}^{\infty}(x, S(y)) = \lim_{n\to\infty} \mathbb{E}\left[\bar{U}^0(X^n_{\tau^n_{n+1}}, Z_{\tau^n_{n+1}})\right]$$

where $(X^n, \tau^n_{n+1}) \in \mathcal{M}^{\perp}(x,y) \times \mathcal{T}$ are ε-optimal strategies.

Suppose that for all $y > 0$, $\mu(y) \leq 0$, then

$$V(x,y) = U(x+y)$$

Idea of the proof :

$$\begin{aligned}\bar{U}^{\infty}(x,S(y)) &= \sum_{i=1}^{n+1}\mathbb{E}\left[\bar{U}^{2(n-i+1)+1}(X^n_{\tau^n_{i-1}},Z_{\tau^n_{i-1}})-\bar{U}^{2(n-i+1)}(X^n_{\tau^n_{i-1}},Z_{\tau^n_i})\right]\\ &+\sum_{i=1}^{n}\mathbb{E}\left[\bar{U}^{2(n-i+1)}(X^n_{\tau^n_{i-1}},Z_{\tau^n_i})-\bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i},Z_{\tau^n_i})\right]\\ &+\mathbb{E}\left[\bar{U}^{0}(X^n_{\tau^n_n},Z_{\tau^n_{n+1}})\right]+\varepsilon_n\end{aligned}$$

An existence result

We assume $\exists N > 0$ such that $\forall n \geq N \qquad \bar{U}^{\infty} = \bar{U}_n$

- This assumption is realistic since our problem with a power and positive utility function could be obtained with an N equal to 2.
- The optimal rules X^N and τ^N_{N+1} are optimal strategies. That is to say,

$$V(x,y) = \mathbb{E}\left[U\left(X^N_{\tau^N_{N+1}} + Y^y_{\tau^N_{N+1}}\right)\right]$$

Our results are consistent with those obtained by Hobson and Henderson for a power utility function but we generalize their work in several ways.

- We use a more general diffusion for the indivisible asset Y.
- Our problem considers a more general utility function.
- We provide a new methodology to solve this problem.

What we have to do now :

- We have to check the case a of non positive utility function.