Unified Multi-name Credit-Equity Modeling: A Multivariate Time Change Approach

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Joint work with: Vadim Linetsky

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- For the two-firm case, we obtain analytical solutions for credit derivatives and equity derivatives, such as basket options, in terms of eigenfunction expansions associated with the relevant subordinated semigroups.

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- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

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  - B^i are n independent standard Brownian motions.

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where ν_i is the Lévy measure of the one-dimensional subordinator $\mathcal{T}^i$ ($\nu_i(A) = \nu(\mathbb{R}_+ \times \ldots \times A \times \ldots \mathbb{R}_+)$ with A in the ith place, for any Borel set $A \subset \mathbb{R}_+$ bounded away from zero),

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    2. the constant ρ_i is:

    $$\rho_i = r - q_i + \phi_i(-\mu_i),$$

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▸ Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

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  - Then $\mathbf{H} = L^2((0,\infty)^n, m)$ is defined on the product space $(0,\infty)^n = (0,\infty) \times ... \times (0,\infty)$ with the product speed density $m(\mathbf{x}) = m_1(x_1)...m_n(x_n)$ and the inner product

$$(f,g)_m = \int_{(0,\infty)^n} f(\mathbf{x})g(\mathbf{x})m(\mathbf{x})d\mathbf{x}$$

Spectral Decomposition (II)

- In the special case when each infinitesimal generator $\mathcal{G}_i$ has a purely discrete spectrum with eigenvalues $\{-\lambda_k^i\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\varphi_k^i(x_i)$,

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and the expansion coefficients are

$$c^f_{\mathbf{k}} = (f,\varphi_{\mathbf{k}})_m, \quad \mathbf{k}\in\mathbb{N}^n.$$

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&= \int_{\mathbb{R}_+^n} \left(\sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\langle\lambda,\mathbf{s}\rangle} c_{\mathbf{k}}^f \varphi_{\mathbf{k}}\right) \pi_t(d\mathbf{s}) && \left(\begin{array}{c}\textit{Spectral representation} \\ \textit{of the} \\ \text{n}-\textit{parameter semigroup}\end{array}\right)
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- Remark: When $n = 1$ the modeling framework is reduced to the Credit-Equity Model of Mendoza-Arriaga et al. (2009).

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases} , \quad i = 1, 2$$

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$$dX_t = [\mu + k(X_t)]X_t\, dt + \sigma(X_t) X_t\, dB_t, \quad X_0 = x > 0$$

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$a > 0 \quad \Rightarrow$ volatility scale parameter (fixing ATM volatility)
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For $c = 0$ and $b = 0$ the JDCEV reduces to the standard CEV process

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The model is consistent with:
leverage effect $\Rrightarrow S \Downarrow \rightarrow \sigma(S) \Uparrow$

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases} , \quad i = 1,2$$

- Let $X_t^i \quad i = 1,2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

$$\underline{\underline{dX_t = [\mu + k(X_t)]X_t\, dt + \sigma(X_t)X_t\, dB_t}}, \quad X_0 = x > 0$$

$$\underline{\sigma(X) = aX^{\beta}} \qquad\qquad \underline{k(X) = b + c\,\sigma^2(X)}$$

CEV Volatility (Power function of X) | Killing Rate (Affine function of Variance)

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The model is consistent with:

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stock volatility–credit spreads *linkage* $\Rrightarrow \sigma(S) \Uparrow \leftrightarrow k(S) \Uparrow$

JDCEV Eigenvalues and Eigenfunctions

- When $mu + b \neq 0$, the spectrum is purely discrete. When $mu + b < 0$, the eigenvalues and eigenfunctions are:

$$\lambda_n = \omega(n-1)+\lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu + b|}{\Gamma(\nu + n)}}\, x\, L^{\nu}_{n-1}(Ax^{-2\beta}), \quad n = 1, 2, \ldots,$$

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- The principal eigenvalue λ_1, A, ν and ω are,

$$\lambda_1 := |\mu|, \quad A := \frac{|\mu+b|}{a^2|\beta|}, \quad \nu := \frac{1+2c}{2|\beta|}, \quad \omega := 2|\beta(\mu+b)|, ,$$

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$$m(x) = \frac{2}{a^2}x^{2c-2-2\beta}e^{-Ax^{-2\beta}}$$

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- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

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The expansion coefficients are given by:

$$c_n^k = (\varphi_n, 1)_m = \frac{A_k^{\frac{1-2c_k}{4|\beta_k|}} (1/(2|\beta_k|))_{n-1} \Gamma(c_k/|\beta_k| + 1)}{\sqrt{(n-1)!|\mu_k + b_k|\Gamma(\nu_k + n)}}, \quad k = 1, 2, \quad n = 1, 2, ...,$$

where $(z)_n = z(z-1)...(z-n-1)$ is the Pochhammer symbol.

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$\phi_1(u) := \phi(u, 0)$, and $\phi_2(u) := \phi(0, u)$ are the Laplace exponents of the marginal one-dimensional subordinators $\mathcal{T}^k$, $k \in \{1, 2\}$, respectively.

Default Correlation

- The default correlation has the form:

$$\text{Corr}\left(\mathbf{1}_{\{\tau_1>t\}},\mathbf{1}_{\{\tau_2>t\}}\right) = \frac{\mathbb{Q}\big(\tau_{\{1,2\}}>t\big)-\mathbb{Q}(\tau_1>t)\mathbb{Q}(\tau_2>t)}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}$$

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$\Rightarrow$ That is, the coordinates $\mathcal{T}^1$ and $\mathcal{T}^2$ of the two-dimensional subordinator are independent.

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▸ Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

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  - **We obtained explicit analytical solutions for all these claims.** Solutions

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- For this example the two diffusion processes X are taken to be JDCEV with the same set of parameters are,

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50	10	0.01	0.5	0	-1	-0.3	0.05

Table: *JDCEV parameter values.*

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- The volatility scale parameter a in the local volatility function $\sigma(x) = ax^{\beta}$ is selected so that $\sigma(50) = 0.2$.

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- The two-dimensional subordinator $\mathcal{T}$ is constructed from three independent Inverse Gaussian processes subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

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- Since the drift is zero ($\gamma = \mathbf{0}$) then the time changed processes $X^i_{\mathcal{T}_t^i}$ are pure jump processes

Numerical Illustration: Survival Probability

- As the sock price falls, the firm's survival probability decreases

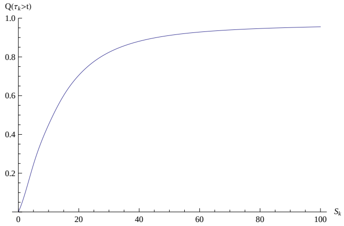
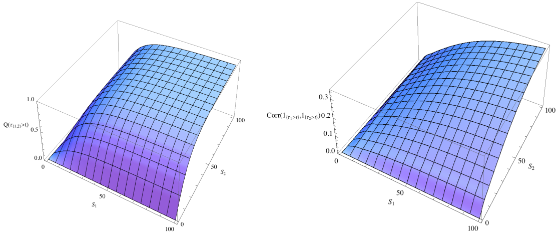


Figure: *Single-name survival probability* $\mathbb{Q}(\tau > t)$ *for* $t = 1$ *year as a function of the stock price* $S_0 = x$.

Numerical Illustration: Joint Survival Probability & Default Correlation

- As the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease



(a) Joint survival probability. (b) Correlation of default indicators.

Figure: *Joint survival probability* $\mathbb{Q}(\tau_{\{1,2\}} > t)$ *and default correlation* $Corr(\mathbf{1}_{\{\tau_1 > t\}}, \mathbf{1}_{\{\tau_2 > t\}})$ *for* $t = 1$ *year as functions of stock prices* S_0^1 *and* S_0^2.

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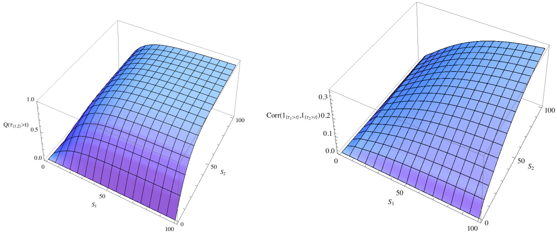

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- When the stock price is relatively high, the default can only be triggered by a large catastrophic jump to zero $\Rrightarrow$ the systematic component $\mathcal{S}^3$ governs large jumps.

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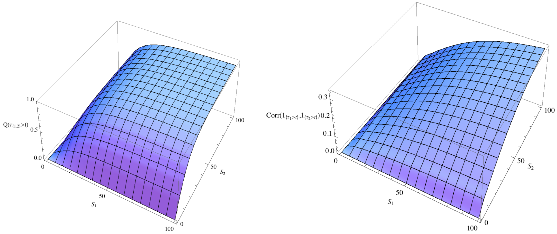

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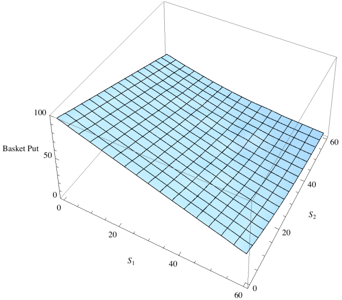
- When the stock price is relatively high, the default can only be triggered by a large catastrophic jump to zero $\Rightarrow$ the systematic component $\mathcal{S}^3$ governs large jumps.
- When the stock price is low, a smaller jump is enough to trigger default $\Rightarrow$ the idiosyncratic components $\mathcal{S}^1$ and $\mathcal{S}^2$ primarily govern small jumps.

Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 100$ as a function of the initial stock prices S_0^1 and S_0^2.



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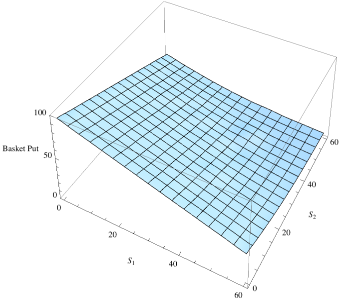

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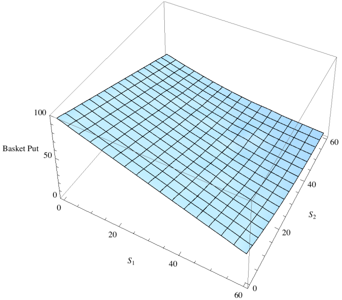

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- Thank you!

Multiparameter Semigroup

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- In our case, the expectation operators associated with the Markov processes X^i define the corresponding semigroups $\{\mathcal{P}^i_{t_i}, t_i \geq 0\}$,

$$\mathcal{P}^i_{t_i} f(x_i) := \mathbb{E}_{x_i}[\mathbf{1}_{\{\zeta_i > t_i\}} f(X^i_{t_i})], \quad x_i \in E_i, \quad t_i \geq 0,$$

in Banach spaces of bounded Borel measurable functions on E_i.

Return

Return

Two Firms Basket Put Option

Return

- The embedded multi-name credit derivative with the notional amount equal to the strike price K and paid at maturity if both firms default

$$e^{-rt}\mathbb{E}[K\mathbf{1}_{\{\tau_1 \vee \tau_2 \leq t\}}] = e^{-rt}K(1 + \mathbb{Q}(\tau_{\{1,2\}} > t) - \mathbb{Q}(\tau_1 > t) - \mathbb{Q}(\tau_2 > t))$$

where the joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$ and marginal survival probabilities $\mathbb{Q}(\tau_k > t)$, $k = 1, 2$; were given earlier.

Two Firms Basket Put Option

Return

- The basket put that delivers the payoff if and only if both firms survive to maturity

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$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+\right]$$

$$= e^{-r\,t} \sum_{n_1,n_2=1}^{\infty} \overbrace{e^{-\phi(\lambda_{n_1}^1,\lambda_{n_2}^2)\,t}}^{\text{2D Lévy Exp.}} c_{n_1,n_2}(K)\varphi_{n_1}^1(x_1)\varphi_{n_1}^2(x_2)$$

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- Where the expansion coefficient $c_{n_1,n_2}(K)$ is given by,

$$c_{n_1,n_2}(K) = \Big((K - w_1 x_1 - w_2 x_2)^+, \varphi_{\mathbf{n}}(\mathbf{x})\Big)_{\mathbf{m}}$$

$$= \int_{\mathbb{R}_+^2} (K - w_1 x_1 - w_2 x_2)^+ \varphi_{n_1}^1(x_1)\varphi_{n_2}^2(x_2) m_1(x_1) m_2(x_2) dx_1 dx_2$$

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$$= K\prod_{k=1}^{2}\left(\sqrt{\frac{\Gamma(\nu_k+n_k)}{\Gamma(n_k)|\mu_k+b_k|}}\,\frac{2|\beta_k|A_k^{\frac{\nu_k}{2}+1}\tilde{K}_k^{2c_k-2\beta_k}}{\Gamma(\nu_k+1)}\right)$$

$$\times\sum_{p_1,p_2=0}^{\infty}\frac{(-1)^{p_1+p_2}\,(\nu_1+n_1)_{p_1}\,(\nu_2+n_2)_{p_2}}{(\nu_1+1)_{p_1}\;p_1!\,(\nu_2+1)_{p_2}\;p_2!}\left(A_1\tilde{K}_1^{-2\beta_1}\right)^{p_1} d\left(A_2\tilde{K}_2^{-2\beta_2}\right)^{p_2}$$

$$\times\frac{\Gamma\left(2c_1-2\beta_1(p_1+1)\right)\Gamma\left(2c_2-2\beta_2(p_2+1)\right)}{\Gamma\left(2c_1-2\beta_1(p_1+1)+2c_2-2\beta_2(p_2+1)+2\right)}.$$

where $\tilde{K}_k = e^{-\rho_k t}K/w_k$.

Two Firms Basket Put Option

Return

- The single-name put on the stock S^k that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_k>t\}}(K-w_k S_t^k)^+\right] = e^{-rt}\sum_{n=1}^{\infty} \overbrace{e^{-\phi_k(\lambda_n^k)\,t}}^{\text{1D Lévy Exp.}} p_n^k(K)\varphi_n^k(x_k),$$

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$$\left\{\frac{1}{(1+c_k/|\beta_k|)}\,{}_2F_2\left(\begin{matrix}\nu_k+n, & \nu_k+1-\frac{1}{2|\beta_k|} \\ \nu_k+1, & \nu_k+2-\frac{1}{2|\beta_k|}\end{matrix}\;;-A_k\tilde{K}_k^{-2\beta_k}\right)\right.$$

$$\left.-\frac{1}{(\nu_k+1)}\,{}_1F_1\left(\begin{matrix}\nu_k+n \\ \nu_k+2\end{matrix}\;;-A_k\tilde{K}_k^{-2\beta_k}\right)\right\},$$

where ${}_1F_1$ and ${}_2F_2$ are the Kummer confluent hypergeometric function and the generalized hypergeometric function, respectively; and $\tilde{K}_k = e^{-\rho_k t}K/w_k$.

Two Firms Basket Put Option

Return

- The single-name put on the stock S^1 that delivers the payoff if and only if both firms survive:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K-w_1S_t^1)^+\right] = e^{-rt}\sum_{n_1,n_2=1}^{\infty}\overbrace{e^{-\phi(\lambda_{n_1}^1,\lambda_{n_2}^2)\,t}}^{\text{2D Lévy Exp.}}p_{n_1}^1(K)c_{n_2}^2\varphi_{n_1}^1(x_1)\varphi_{n_2}^2(x_2),$$

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- where c_n^2 are the coefficients of the expansion for the survival probability of the second stock and,
- $p_n^1(K)$ are the expansion coefficients for the single-name put on the first stock.

Return

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